# Roots of P'(z) lie in the convex hull of the roots of P(z)

Andrew • 17 Sep 2026

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I was recently asked (by someone who didn't do math) what makes math beautiful. I think this is a very difficult question.

One possible answer, which another person gave, is that it's the connections between different parts of math which you never thought would be related. To me this slightly misses the mark. I think this makes doing math enjoyable, surprising, exciting, etc. But I feel that the reason we are often surprised or amazed by certain connections is that we already have certain ideas of what this or that concept/object/phenomenon is about. The connection then tells us that we ought to be thinking of this thing in a different way. The whole value or "beauty" seems to rely too much on what is in our heads, instead of what the mathematics really is. And I would like to point to something intrinsic to math when I try to explain what makes it beautiful, and not to our conceptions of it.

The answer I want to propose is that often things are larger than the sum of their parts. For example, a theorem is something more than just the statement and a chain of syllogisms supporting it, even though technically, that is all there is. This is kind of like how a painting is something more than just all its brush strokes, even though that is all there really is. One may wonder whether there really is such a thing as a "painting" beyond just its set of brush strokes, and similarly for the theorem, or whether this is once again just a human-imposed notion. I will not get into this metaphysical question here.

One example of this phenomenon is the way that some proofs of theorems involve a reduction step. The most common instance is induction: we reduce to the case, say, $n = 0$ or $n = 1$. Another example might be reducing to the affine case in algebraic geometry, or in general any kind of global-to-local reduction (e.g. reducing to the $\mathbb{R}^n$ case in manifolds). It often seems that the reduction step is not much at all, and the thing that we've reduced to isn't either, but somehow the original theorem is something substantial.

One particular example is the following proof of the statement in the title, due to Gauss. I should note that I think the statement itself is very beautiful as well: there's something about how it's so simple and so geometrically striking that appeals to me. Also, I somehow hadn't heard or noticed this fact until today, even though I have lots of familiarity with each of these objects.

The reduction step is the following. We claim that it suffices to show that if the roots of $P(z)$ all lie in the closed upper half plane, then so do the roots of $P'(z)$. The reason is the following:

**Lemma:** The convex hull of $S$ (say, when $S$ is a finite set of points) is the intersection of all the closed half-planes containing $S$.

*Proof:* This is straightforward in barycentric coordinates.

Let me also give a more geometric proof of the Lemma. The key geometric idea is that there is always a line separating compact convex sets. To see this, pick two points $a$ and $b$ in the two sets minimizing the distance between $a$ and $b$. These exist by compactness. Then consider the perpendicular bisector of the segment connecting $a$ and $b$. If, say, there existed $b'$ in the second set on the same side of the line as $a$, then we can drop the altitude from $a$ to the segment $bb'$, which by convexity lies in the second set. The foot is then closer to $a$ than $b$ is, contradiction.

Returning to the Lemma, one inclusion is clear: the intersection of all the halfplanes is convex and contains our given set of points, so it contains the convex hull. Now suppose there existed a point outside the convex hull but inside the intersection of the halfplanes. Since both the point and the convex hull are convex and compact, we can find a hyperplane separating them. But this implies the point isn't in the intersection of all halfplanes containing the convex hull, let alone the original set. $\square$

Returning to our original statement, we can do the reduction by considering all the half-planes containing the roots of $P(z)$ and doing the appropriate transformation to send them to the upper half plane $\{\mathrm{Im}(z) \geq 0\}$.

But now it is just a calculation. If $z$ is a root of $P'(z)$ then in particular $\frac{P'(z)}{P(z)} = 0$. If the roots of $P$ are $z_1, \ldots, z_k$, then this is $\sum \frac{1}{z-z_i}$. If $z$ had negative imaginary part, then so would $z - z_i$, which would imply $\frac{1}{z-z_i}$ has positive imaginary part, so they could never sum to $0$.

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I should note that there is a purely algebraic proof on wikipedia, which is not nearly as beautiful.