# Laplace equation

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One of the great things about mathematics (which I feel is overlooked more and more nowadays) is the feeling of mystery and suspense. The thing about mystery and suspense isn't just that you don't know what's going to happen next. It's that you can't predict what's going to happen next. It has to be something that's unexpected.

Here is one such example. Consider a discrete analog of the Laplace equation with Dirichlet boundary conditions. Namely, suppose we have a finite connected subset $A$ of the integer lattice $\mathbb{Z}^2$, with boundary $\partial A$. The Laplacian of a function on $\overline{A} := A \cup \partial A$ is defined on $A$ and is given by the average value of the function on the 4 neighboring points, minus the value of the function at the point.

As usual, the problem is to find a way of extending a real-valued function $F$ on the boundary of $A$ to the interior such that its Laplacian vanishes.

The solution is to consider a random walk (!) starting at a point $x \in \overline{A}$. Since the set $A$ is finite, it's eventually going to hit $\partial A$. Take the value of $f$ at $x$ to be the expected value of $F$ at the first point where it hits $\partial A$.

The fact that this works is almost immediate: by doing casework on the first step, the value of $f$ at a point is the average of its value at the neighboring points. And on $\partial A$, the random walk stops immediately and so the expected value is just $F(x)$.

As with most good mysteries, the solution is obvious once it's revealed. And yet, it is so unexpected.