Orientability and two-sidedness

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Reference: Weeks "The Shape of Space" sec 8

One of the often overlooked benefits of doing mathematics is that it helps to clarify your thinking.

And I think one of the biggest difficulties that people run into when doing mathematics is that their thinking is not very clear. For example, often when I read textbooks I get confused because I think that what is written is wrong or nonsensical because of XYZ reasoning, but it turns out that XYZ is subtly (or sometimes not so subtly) flawed. I then wind up spending many minutes trying to figure out what is wrong with XYZ.

Let me give one example of a distinction which I think I would not have understood before taking geometry/topology classes, even though it is not something which is too difficult to understand for non-mathematicians. This is the distinction between orientability and two-sidedness (of, say, two-manifolds).

One can think about orientability as the following. If you were a creature living in the space and took a walk and came back to your starting place, you would not be reversed no matter how you walked. One can think of being "reversed" as being, say, flipped in the mirror.

One can think of two-sideness as the following. Suppose the manifold is a two-manifold which is embedded in a three-manifold. If you were a "3D" ant (living in the ambient space) and you were walking along the surface of the manifold, then by doing so you could traverse the entirety of the surface.

The key distinction is that the first one is an intrinsic property (in that it only depends on the manifold itself) and the second one is an extrinsic property (in that it depends on the ambient space and how the manifold is embedded). Note that when we try to define two-sidedness, it doesn't really make sense to talk about sides unless we have an ambient space.

It is often said that "the Mobius band has one side." But this doesn't really make sense, because whether or not the Mobius band has one side depends on the ambient space. In fact, it is possible to embed a Mobius band into a three-manifold such that it has two sides. One can take a cube with top and bottom faces identified (in the usual way) and front and back faces identified with a side-to-side flip. Then the square cross-section which is halfway up the cube has two edges identified with a flip, so is a Mobius band. But if an ant were to walk on the square, there is no way for it to get from the top to the bottom with its legs on the square, so there are indeed two sides.

What should be said instead is “the Mobius band is nonorientable.” This is because the “Mobius band” refers to the space itself, hence can only be described via intrinsic properties.

Of course, the fact that these two are conflated is not accidental: if the ambient space is orientable then they are the same. (This is because the tangent space to the manifold and the normal vector sum to the tangent space to the ambient space, so given an orientation on the ambient space and the manifold this determines a unique unit normal vector. Conversely, given a unit normal vector and an orientation on the ambient space, this determines an orientation on the tangent space to the manifold.)

This allow us to understand what is going on with the “two-sided Mobius band.” The ambient space is non-orientable! Indeed, if you were to walk through the back face, you would come back in the front face mirror-reversed (with a side-to-side flip). And since it is very rare (at least in popular culture) that we deal with non-orientable three-manifolds, it makes sense that these two concepts are conflated.