Convex subsets of R^n

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A lot of (pictoral) topological arguments have a sort of nebulous or cloudy feeling in the sense that the pictures you draw are only up to homeomorphism or homotopy or something. They are almost never precise as drawn. In algebraic geometry they approach diagrams or schematics rather than literal pictures.

Recently I encountered a refreshing example of some very simple and very concrete geometry. In proving Poincare duality one eventually reduces to the $\mathbb{R}^n$ case relative to a compact, convex (and connected) subset. One of the key ideas in this case is that the Cech cohomology of this subset is easier to compute than in the general case. In fact, it's just the same as the singular cohomology.

To prove this, one needs to find a cofinal set of open neighborhoods, all of which have the same cohomology as the subset itself. Of course, it would suffice if these open neighborhoods deformation retracted on to the subset.

Intuitively, for any compact set (not necessarily convex) one can take for this cofinal set a ``tubular neighborhood'' of the boundary, along with the interior. However, it is not so clear, at least without additional theorems about tubular neighborhoods, that it actually deformation retracts onto the set.

There is a very nice concrete construction in the convex case. Consider the set of points at most ε from the subset. (The notion of ``distance to the subset'' is well-defined since the subset is compact.) Now this is clearly cofinal (sketch: the closures of these subsets have intersection precisely the original subset, then if there were a point in an open not contained in any of these, look at the limit point of this sequence as $\varepsilon \to 0$).

The really nice part is the construction of the deformation retraction. We claim that there is a *unique* closest point to the subset, for any point outside it. Suppose that the minimum from a point P was attained twice, via two points $A \neq B$ inside the subset. Then since the subset is convex, the line connecting A and B also lies in the subset. But then forming the isoceles triangle PAB, the altitude from P is strictly shorter than either PA or PB, by the Pythagorean Theorem.

Now we can simply take the deformation retraction given by moving along the line connecting P to the closest point in the subset.