Weak inverse for vector bundles

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One pattern that I've noticed is that I consistently overestimate how difficult/deep statements about vector bundles are. Perhaps it is that they have simple proofs but are genuinely deep results, or perhaps this is just a reflection of my weak intuition. Regardless here is one such result, whose proof I also thought was quite clean.

Suppose $E \to M$ is a (complex) vector bundle over a compact manifold (the proof also works over manifolds with a finite good cover). Then there exists another vector bundle $F \to M$ such that $E \oplus F$ is trivial.

Our goal will be to find a surjection from a trivial vector bundle. Take an open cover $\{U_i\}$ of M on which E is trivial and by compactness assume the cover is finite. By shrinking lemma we can find another cover $\{V_i\}$ on which E is trivial and $\overline{V_i} \subset U_i$.

The key idea is that we can find local sections of E over V_i, because E is trivial. If E is rank k, then we can just take these to be the constants $e_1, \ldots, e_k$. Since the V_i are strictly contained in the U_i, we can extend these by 0 so that they are 0 outside the U_i and constant inside the V_i. Call these (now global) sections $s_{i1}, \ldots, s_{ik}$.

Let $N = |I|$. Putting all these together, we have a surjection $\mathcal{C}^{Nk} \times M \to E$ given by taking the appropriate linear combination of the sections s_{ij} evaluated at the appropriate point. Since at every point we've chosen the sections to span the vector space, this is surjective. Moreover, this is by construction a bundle map.

Hence we have a surjection from a trivial vector bundle to E. Take F to be the kernel. Since every SES of vector bundles splits, we are done.

What I really like about this proof (and what really surprises me about it) is how elementary it is. The statement sounds really mysterious at first but in reality it just comes down to (essentially) the very definition of a vector bundle: it's locally trivial. (I guess at the end we implicitly use the existence of a Riemannian structure to conclude every ses splits, but this is not such a major point in my mind.) And so somehow, out of nothing, we get this statement which really seems quite significant: any vector bundle, no matter how complicated or twisted or whatever, can be made larger via direct sum so that it's trivial.