Cancellation property for etale morphisms

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Reference: Milne etale cohomology (book, not lecture notes) cor 3.6

In my posts I tend to emphasize how much I enjoy geometric arguments and intuition. After all, who doesn't like a good picture? Often I dislike purely algebraic arguments because I feel that they "don't tell me anything about why it's true."

But there are certain times where I find some kind of asthetic beauty in algebra, diagram chasing, etc. In some sense they "dont tell us why it's true." But in some ways this is because it's true purely formally, so there isn't even a need to say "why" it's true. That is, they tell us that there is really no mystery to the statement.

Here is one such example. Say we have $g : Y \to X$, a morphism of S-schemes, with the map $X \to S$ being f. Then if fg is etale and f is unramified then g is etale. In particular if fg is etale and f is etale then g is etale.

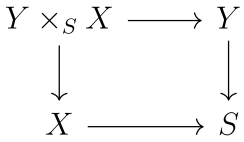
The key fact is that a morphism is unramified iff the embedding of the diagonal is an open immersion.

The key "trick" is to base change the emebedding of the diagonal via g.

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ {\scriptstyle \Gamma_g}\downarrow & & \downarrow{\scriptstyle \Delta} \\ Y \times_S X & \xrightarrow[g\times 1]{} & X \times_S X \end{array}$$

Note that this is cartesian: intuitively, it's the pairs of (y, x_1) and x_2 such that $(g(y), x_1) = (x_2, x_2)$. In other words, it's parameterized by y, with parameterization given by Γ_g.

Since open immersions are etale, the diagonal on the right is etale. Since etale is stable under base change, the graph is therefore etale. But note that g is precisely the composition of the graph and the projection to X. This is etale, because it can once again be viewed as the base change of an etale map:



Since the composition of etale is etale, we are done.

One pattern I am noticing is that considering the graph of a function, while not a very complicated idea, seems to lead to these really elegant proofs (or, what I view as really elegant, idk). Other than this argument, another argument that comes to mind is the proof of the Lefshetz fixed point formula. The key insight there is that fixed points correspond to the intersection of the diagonal and the graph. The rest of the proof is “merely” computations, once you realize this.