Graded-commutativity of cohomology

Andrew • 8 Apr 2026

Last time I wrote about understanding something from two different points of view. This time I once again find myself in a similar situation, comparing two instances of the same phenomenon.

The phenomenon is the following. On both singular cohomology and de Rham cohomology, there is a product structure. (This is important because it can distinguish spaces.) It is graded-commutative in the sense that it satisfies $a \cdot b = (-1)^{|a||b|} b \cdot a$, where $|a|$ and $|b|$ denote the degrees of a and b.

In the de Rham case this is completely natural, because it even holds on the level of forms. The reason it holds on the level of forms is intuitively that swapping two coordinates “reverses orientation.” Integrating $dx_1 \wedge dx_2$ should give the negative of integrating $dx_2 \wedge dx_1$.

In the singular case it is less clear. Tracing through the proof, it ultimately comes from the purely algebraic fact that the differential on the tensor product of complexes is $d \otimes 1 + (-1)^p 1 \otimes d$, and not just the sum of the differentials (so that the differential squared is zero). Then in order to make a certain map a chain map, you need to introduce an extra sign when you swap the two factors. [see e.g. haynes miller notes pg 89, in the diagram right before theorem 29.2]

On the one hand, I find it really nice that we can see the same phenomenon appearing in multiple different places, for seemingly different reasons. Of course it makes us more confident that it really is there. But it also gives me the feeling that there’s something deeper that’s we’ve discovered in the process of doing mathematics.

On the other hand, the contrast between the way it looks in one case and the way it looks in the other highlights some of my gripes with singular (co)homology in general. That is, that it replaces geometric intuition with algebraic intuition, which I don’t find particularly persuasive. I find it most helpful to think about the graded-commutativity as arising from the properties of the wedge product, which in turn arise from the properties of integration. But if you only looked at the singular cohomology, the sign kind of seems like it’s forced upon you, and you don’t really want it there. (I have in mind also other examples of signs, like how the differential on cohomology is the naturally induced one from homology, but *with a sign*.)

But we can also think about it in a more positive light. One of the great benefits of singular cohomology is that it applies to more general spaces than de Rham cohomology (for example, maybe you want to consider wedge products of spaces). What all of this is telling us is that our intuition from the smooth manifold case (about orientations and integration and so forth) carries over to the topological space case, in some sense. The reason is that the graded-commutativity is much less fragile than our smooth-manifold-perspective would have us believe, because it generalizes to the non-smooth (and even wilder topological spaces) setting. It's somehow deeply baked in to how cohomology works.