Pushforward/pullback and restriction/extension of scalars

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One thing I really like about algebraic geometry is that it often gives you two ways of thinking about the same thing, namely the algebraic way and the geometric way. More often than not, I find that the geometry somehow "explains" the algebra, even though in reality they're just the same thing in a different language.

One example is the pushforward/pullback of $\mathcal{O}_X$-modules and restriction/extension of scalars. Suppose we have $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ a morphism of schemes with corresponding maps $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ and $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. Given an $\mathcal{O}_X$-module $\mathscr{F}$ the pushfoward $f_*\mathscr{F}$ is naturally an $\mathscr{O}_Y$-module, where the action is induced by first applying $f^\flat$ and then using naturality. This is what I mean when I say the pushfoward of $\mathcal{O}_X$-modules. On the other hand, given an $\mathcal{O}_Y$-module $\mathscr{G}$, the pullback $f^{-1}\mathscr{G}$ is not naturally an $\mathcal{O}_X$-module, so instead we take $f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathcal{O}_X$. Here $f^{-1}\mathscr{G}$ gets the structure of an $f^{-1}\mathscr{O}_Y$-module by functoriality (with a bit of work) and $\mathcal{O}_X$ also has the structure via $f^\sharp$.

It turns out that in the affine case, the pushfoward is precisely restriction of scalars and the pullback is precisely the extension of scalars. If $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$, $\mathscr{F} = \tilde{M}$, $\mathscr{G} = \tilde{N}$, where M is an R-module and N is an S-module, then $f_*\mathscr{F} = \tilde{M}$ where now we view M as an S module via restriction of scalars. Similarly $f^{-1}\mathscr{G} = \widetilde{N \otimes_S R}$, which is the extension of scalars of N to R.

We expect pullback to be the left adjoint of pushfoward, since the same is true for general sheaves (not necessarily with a module structure). Hence we see that extension of scalars is the left adjoint of restriction of scalars.

Of course, one might say "wait, isn't this backwards?" Presumably, the way you prove pullback is the left adjoint of pushfoward is via reducing to the affine case (well, at least in the case of quasi-coherent $\mathcal{O}_X$ modules…). But my point is just that the intuition, at least for me, for why extension/restriction are adjoints comes from the geometric setting. (Although, I just checked Gortz-Wedhorn and apparently you can prove pullback/pushforward are adjoints directly, and it holds without the quasi-coherent assumption.)

Back when I took commutative algebra, I always felt that I was missing something but wasn't confused about anything in particular. I think what was missing was the geometric intuition. I remember restriction of scalars in particular felt "wrong" and "strange" to me. Now I know it is nothing but pushforward of $\mathcal{O}_X$-modules.