Products

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There are some theorems which are helpful and some theorems which are not. What I mean by “helpful” is that they tell you something about “how to think about X” or “what X really means.” Obviously, some theorems that seem unhelpful at first can actually turn out to be helpful if you interpret them in the right way.

In a similar way, I think there are some ways of defining things which are helpful and some ways which are not. You can recognize the second type when you read the definition and you’re like “okay…?” You can recognize the first one when it feels like you immediately have a sense of what is intended.

I recently read a very simple example of a “helpful” definition. It struck me because it is so simple that I almost never think about trying to define it. The notion is that of the product of two topological spaces, or just two sets in general. Usually one says something like $A \times B = \{(a, b) : a \in A, b \in B\}$ or in simpler terms, the set of ordered pairs, where the first element is from A and the second one is from B.

The definition I read comes from “The Shape of Space” by Weeks. He says:

“A cylinder is the product of a circle and an interval. It qualifies as such because it is both

1. A bunch of intervals arranged in a circle, and
2. A bunch of circles arranged (in this case stacked) in an interval.”

He then goes on to give another example:

“A torus is a second example of a product. It’s the product of one circle (drawn dark in Figure 6.2) with another (drawn light). This is because the torus is a circle of circles in two different ways: both as a dark circle of light circles and as a light circle of dark circles.”

First off, I think this definition is extraordinarily clear. I can imagine many high schoolers having trouble with the notion of ordered pairs, but I can’t imagine many having trouble with this definition, especially with the pictures.

Second, this definition hints at the fact that the product has two natural projection maps. Namely, when you have “a bunch of intervals arranged in a circle” you can collapse the intervals and just get the circle. Moreover, each of those projection maps is a (trivial) fiber bundle, because they’re “stacked.” In fact, this is the defining property of the product: if you have a general fiber bundle, then it is a product bundle if there are natural fiber bundles over not just the base, but also the fiber.

So I guess that is what makes a definition “helpful” to me. First, it should be clear even to someone without a lot of knowledge of technical vocabulary. Second, it should hint at some kind of essential property.