Symmetry of Tor

Andrew • 18 Mar 2026

One thing I like about doing mathematics is the feeling of surprise when you connect things from two really different places.

It was implied in class today that Tor is symmetric. After thinking about it a little bit, I decided to look it up. I didn't get much that I understood, but I did see the words "double complex."

And that's when I had an idea. Suppose we want to compute $\mathrm{Tor}_i^R(M, N)$. We can if take two resolutions, one of M, and one of N, tensor them with the other one, and then join the chain complexes together at their endpoint. There's a natural way to fill in the rest of the square, namely with the tensor products of the resolving free modules. But since these are free, the rows and columns are exact (except for the first one).

And then I remembered the general principle: if the rows and columns of a double complex are exact, then the (co)homology of the first row and the first column are the same. This is exactly the same principle as in the isomorphism between the de Rham and Cech cohomologies. In both cases, you fill in a kind of "interpolating" double complex.

Intuitively, I think what is going on here is that the "interior" of the double complex "doesn't add any (co)homology" and so the (co)homology can be "transfered" from one part of the "boundary" to the other. But what I think is really cool is that exactly the same statement (well, up to duality) applies in both cases, which on the surface seem pretty different. In the de Rham-Cech case, we have these two very geometric cohomology theories and the double complex literally interpolates between them (by changing the presheaf to be the p-forms). In the Tor case, well, I guess we are interpolating between resolutions (?).

Anyway I thought the de Rham-Cech thing was super cool when I first saw it and it's even cooler that it has come up again.