Distinct Distances Beyond the Plane

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https://functor.network/user/3333/entry/2106

In an August post, I discussed the theorem of Jonathan Tidor, Hung-Hsun Hans Yu and Dmitrii Zakharov on distinct distances in three-dimensional space. I ended with two questions: can the small loss in their bound be removed, and what happens in higher dimensions? A preprint dated September 23, 2026, by OpenAI (OpenAI 2026) answers both questions.

Let us recall the problem. For a finite set $S \subset \mathbb{R}^d$, its *distance set* is

$$D(S) = \{\|x-y\| : x, y \in S,\ x \neq y\},$$

where $\|x-y\|$ is the usual Euclidean distance. Repeated occurrences of the same distance contribute only one element to $D(S)$.

How few distinct distances can N points determine? Formally, define

$$f_d(N) = \min_{\substack{S \subset \mathbb{R}^d \\ |S|=N}} |D(S)|.$$

Erdős introduced the distinct distances problem in 1946 (Erdős 1946). Its difficulty comes from the freedom to arrange the points in any configuration, including highly structured ones in which many pairs have the same distance.

The natural construction is an integer grid. Suppose first that $N = m^d$ and take

$$S = \{1, \dots, m\}^d.$$

For two distinct grid points x and y, the squared distance is

$$\|x-y\|^2 = (x_1-y_1)^2 + \cdots + (x_d-y_d)^2.$$

This is an integer between 1 and $d(m-1)^2$. Since different positive distances have different squares, we obtain

$$|D(S)| \le d(m-1)^2 \le dN^{2/d}.$$

For an arbitrary N, choose N points from the grid with $m = \lceil N^{1/d} \rceil$. The same argument gives

$$f_d(N) \le dN^{2/d}.$$

Thus a grid produces many repeated distances. Could a more ingenious arrangement produce substantially fewer?

For each fixed dimension $d \geq 3$, the higher-dimensional Erdős distinct distances conjecture predicted that the grid has the correct order of growth. In 2008, Solymosi and Vu (Solymosi and Vu 2008) obtained lower bounds approaching the predicted exponent $2/d$. In August 2026, Tidor, Yu and Zakharov (Tidor et al. 2026) reached

$$|D(S)| \geq N^{2/3-o(1)}$$

in dimension three.

The new theorem gives the full constant-factor bound in every higher dimension.

> **Theorem.** For every integer $d \geq 3$, there exists a constant $c_d > 0$ such that every set S of $N \geq 2$ distinct points in $\mathbb{R}^d$ satisfies
>
> $$|D(S)| \geq c_d N^{2/d}.$$

The constant may depend on the dimension, but not on the number or arrangement of the points. There are no assumptions about spacing, general position, or how strongly the points concentrate on lines, planes, or other surfaces.

Combining the theorem with the grid construction gives

$$c_d N^{2/d} \leq f_d(N) \leq d N^{2/d}.$$

Hence the minimum number of distinct distances has order $N^{2/d}$ for every fixed $d \geq 3$. The exponents in dimensions three, four and five are therefore $2/3$, $1/2$ and $2/5$, respectively.

Why does replacing $N^{2/3-o(1)}$ by $cN^{2/3}$ matter? An exponent approaching $2/3$ can still conceal an unbounded loss. For example,

$$\frac{N^{2/3}}{\log N} = N^{2/3-o(1)},$$

but its ratio to $N^{2/3}$ tends to zero. The new theorem prevents even an arbitrarily slowly growing loss of this kind.

The restriction to dimensions at least three is essential. In the plane, the square grid determines only order $N/\sqrt{\log N}$ distinct distances, so a lower bound proportional to N is impossible. The celebrated theorem of Guth and Katz (Guth and Katz 2015), first announced in 2010 and published in 2015, gives

$$f_2(N) \geq c\frac{N}{\log N}.$$

The planar conjecture predicts the stronger lower bound proportional to $N/\sqrt{\log N}$. Thus the higher-dimensional theorem settles a different asymptotic question from the remaining planar problem.

The proof develops the connection between repeated distances and rigid motions. A rigid motion is a transformation preserving Euclidean distances. Two segments of the same length can be carried onto one another by such a transformation, so repetitions of distances can be studied through families of motions.

In suitable algebraic coordinates, these families are represented by affine subspaces in a larger parameter space. The problem becomes one of controlling their intersections using polynomials.

The main obstacle is concentration: many of the subspaces may lie inside sets defined by polynomial equations of relatively small degree. The proof controls this behavior across several degree scales. This is crucial for obtaining a fixed positive constant rather than a bound with a small loss in the exponent.

After these geometric estimates are established, a random sampling argument produces the final contradiction. A polynomial chosen to vanish on a small sample of subspaces would be forced, by their numerous intersections, to vanish on more subspaces than the polynomial estimates permit.

There is also a connection with Falconer's distance conjecture, discussed recently in this blog. Both problems ask how the size of a set constrains the size of its distance set. Here size means the number of points and distances; in Falconer's problem, it is measured using Hausdorff dimension and Lebesgue measure.

For finite point sets in every dimension above the plane, the conclusion is now sharp up to a constant factor: the integer grid predicts exactly the right order.

References

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