On The Optimal Covering Density

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In two preprints dated September 23, 2026, OpenAI (OpenAI 2026a, 2026b) determined the optimal order of the worst-case density needed to cover high-dimensional space by translates of a convex body. The answer is proportional to $d \log d$, whether the translates must follow a lattice or may be placed freely.

In a previous post, we discussed sphere packing: arranging equal balls without overlaps while occupying as much space as possible. Covering asks a related question. We now require every point of space to belong to at least one copy of our shape, and overlaps are allowed. How economically can this be done?

Let $K \subset \mathbb{R}^d$ be a *convex body*: a compact set with nonempty interior that contains the line segment joining any two of its points. Balls, cubes and simplices are examples.

For $x \in \mathbb{R}^d$, the translate

$$K + x = \{k + x : k \in K\}$$

is a copy of K moved by the vector x. Throughout this post, copies retain the same size and orientation.

A set X of translation vectors gives a covering if

$$K + X = \bigcup_{x \in X} (K + x) = \mathbb{R}^d.$$

We require X to be locally finite, meaning that every bounded region contains only finitely many of its points.

Since a covering occupies all of space, the fraction of space covered is always 1. To measure efficiency, we instead count the total volume of the copies used, including overlaps. More precisely, define

$$\delta(K, X) = \mathrm{Vol}_d(K) \limsup_{R \to \infty} \frac{|X \cap [-R, R]^d|}{(2R)^d},$$

where Vol_d denotes d-dimensional volume and $|\cdot|$ denotes the number of points. The optimal translative covering density is

$$\theta_{\mathrm{tr}}(K) = \inf_{\substack{X \text{ locally finite} \\ K + X = \mathbb{R}^d}} \delta(K, X).$$

Every covering has density at least 1. For example, the unit cube $[0,1]^d$, translated by the vectors in $\mathbb{Z}^d$, covers space with density exactly 1.

A particularly structured covering uses a *lattice*

$$L = \{m_1 v_1 + \cdots + m_d v_d : m_1, \ldots, m_d \in \mathbb{Z}\},$$

where $v_1, \ldots, v_d$ are linearly independent vectors. Its fundamental parallelepiped has volume

$$\det L = |\det(v_1, \ldots, v_d)|,$$

called the covolume of the lattice.

There is one translation vector per fundamental parallelepiped, so a lattice covering has density

$$\frac{\mathrm{Vol}_d(K)}{\det L}.$$

Consequently, the optimal lattice covering density is

$$\theta_{\mathrm{lat}}(K) = \inf_{\substack{L \text{ a full-rank lattice} \\ K + L = \mathbb{R}^d}} \frac{\mathrm{Vol}_d(K)}{\det L}.$$

Clearly,

$$\theta_{\mathrm{tr}}(K) \le \theta_{\mathrm{lat}}(K),$$

because lattice coverings are among the allowed translative coverings.

The history contains an important distinction between these two problems. In 1957, Rogers (Rogers 1957) proved that every convex body in dimension $d \ge 3$ admits a translative covering of density at most

$$d \log d + d \log \log d + 5d.$$

Here and below, logarithms are natural. Such a covering can be periodic, with its translation vectors forming a finite union of translates of a lattice. A finite union of this kind need not itself be a lattice.

For a single lattice, Rogers's 1959 bound (Rogers 1959) was much larger, growing faster than every fixed power of d. More than sixty years passed before a polynomial bound was obtained. In work first posted in 2020 and published in 2022, Ordentlich, Regev and Weiss (Ordentlich et al. 2022) proved

$$\theta_{\mathrm{lat}}(K) \le C d^2$$

for an absolute constant C.

In 2026, Li and Liu (Li and Liu 2026) brought the upper bound within an iterated-logarithmic factor of $d \log d$. They also constructed origin-symmetric bodies requiring lattice covering density at least $cd \log d$.

The two OpenAI papers complete the picture.

> **Theorem.** There are absolute constants $c, C > 0$ such that every convex body $K \subset \mathbb{R}^d$, $d \geq 2$, satisfies
>
> $$\theta_{\mathrm{tr}}(K) \leq \theta_{\mathrm{lat}}(K) \leq Cd \log d.$$
>
> For every sufficiently large d, there is an origin-symmetric convex body $K_d \subset \mathbb{R}^d$ such that
>
> $$\theta_{\mathrm{tr}}(K_d) \geq cd \log d.$$

Origin symmetry means that $K_d = -K_d$. Thus, even among symmetric bodies, some shapes force substantial overlap in every covering.

The constants are independent of both the dimension and the body. The body K_d in the lower bound may depend on d. This is a worst-case result: cubes still have optimal covering density 1 in every dimension.

The upper bound (OpenAI 2026a) develops Li and Liu's method of splitting the coordinates into a large horizontal part and a much smaller vertical part. Suitable horizontal sections of the body can be covered almost completely by one common horizontal lattice. Vertical translations bring several such sections into play at each height.

The difficulty is to coordinate these contributions while keeping all translation vectors in one lattice. The proof organizes successive binary choices into a hierarchy and combines them through a single linear shear. Controlling the accumulated losses across this hierarchy removes the remaining iterated-logarithmic factor.

The lower bound (OpenAI 2026b) uses a different probabilistic construction. Start with a ball and intersect it with finitely many randomly chosen symmetric slabs. The resulting body remains convex and origin-symmetric.

With positive probability, the body retains substantial volume while defeating every covering of density below $cd \log d$. To handle the enormous collection of possible arrangements, the proof localizes and rounds their translation vectors to obtain finitely many patterns. Strong probability estimates show that all these patterns leave holes.

A periodization argument then extends the obstruction to arbitrary translative coverings: an efficient unrestricted covering would yield a periodic covering of nearly the same density.

Together, the results identify the precise order of the largest optimal covering density in dimension d. The hardest convex bodies require overlap of order $d \log d$, and a lattice always suffices to achieve this order. Complete freedom in placing the copies therefore does not change the asymptotic order of the worst-case problem.

References

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