No Periodic Tiling with one 3D Tile

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In a preprint dated September 23, 2026, OpenAI (OpenAI 2026) constructed a single tile that fills three-dimensional space by translations but admits no periodic translational tiling. For the integer-lattice version of the problem, three is the smallest possible dimension.

The question is easy to describe. Suppose that identical copies of one tile can fill space without gaps or overlaps. Must there be a way to arrange them in a repeating pattern?

The restriction to translations is important. We may slide a copy of the tile, but every copy must retain the same orientation.

This differs from the problem solved by the famous planar “hat” tile of Smith, Myers, Kaplan and Goodman-Strauss (Smith et al. 2024), where rotated and reflected copies are allowed.

Let us first formulate the problem on the integer lattice.

Write $\mathbb{Z}^d$ for the set of points in $\mathbb{R}^d$ with integer coordinates. A finite nonempty set

$$F \subseteq \mathbb{Z}^d$$

is a *translational tile* if there is a set of translation vectors $A \subseteq \mathbb{Z}^d$ such that the sets

$$F + a = \{f + a : f \in F\}, \qquad a \in A,$$

partition $\mathbb{Z}^d$.

Thus every point $z \in \mathbb{Z}^d$ must have exactly one representation

$$z = f + a, \qquad f \in F, \quad a \in A.$$

The translates cover every lattice point, and distinct translates do not overlap.

For example, take

$$F_0 = \{0,1\}^3, \qquad A_0 = 2\mathbb{Z}^3.$$

Each integer coordinate has a unique expression as an even integer plus either zero or one. Applying this in all three coordinates shows that F_0 tiles $\mathbb{Z}^3$ with translation set A_0.

This tiling repeats after translation by each of

$$(2,0,0), \qquad (0,2,0), \qquad (0,0,2).$$

More generally, a tiling in $\mathbb{Z}^d$ is called *periodic* if there are d linearly independent integer vectors $v_1, \ldots, v_d$ such that

$$A + v_i = A \qquad (i = 1, \ldots, d).$$

Integer combinations of these vectors are also periods. The whole tiling can therefore be generated by repeating a finite pattern in every dimension.

Throughout this post, periodicity means repetition in all d independent directions. In dimension three, having just one period is a weaker property.

The *periodic tiling conjecture* predicted that whenever a finite set F tiles $\mathbb{Z}^d$, it admits a periodic tiling.

Notice the quantifiers. The conjecture asks whether at least one periodic arrangement exists. A tile may have both periodic and nonperiodic arrangements, so finding an irregular tiling alone does not refute it.

There were strong reasons to expect a positive answer in small dimensions.

The one-dimensional case is classical. For dimension two, Bhattacharya (Bhattacharya 2020), in a paper published in 2020, proved that every finite subset of $\mathbb{Z}^2$ that tiles the integer lattice also admits a periodic tiling. The finite set may be disconnected and may have holes.

Greenfeld and Tao (Greenfeld and Tao 2024) overturned the general conjecture in work announced in 2022 and published in 2024. They constructed counterexamples in sufficiently high dimensions.

The new theorem brings this failure down to dimension three. It also gives a Euclidean tile made from finitely many unit cubes.

For a finite set $F \subseteq \mathbb{Z}^3$, define

$$K = \bigcup_{f \in F} \left(f + [0,1]^3 \right).$$

We replace each lattice point of F by the unit cube having that point as its lower corner. The resulting set may have several components; all of them are translated together as one prescribed tile.

For tilings of $\mathbb{R}^3$, we allow overlaps on sets of zero three-dimensional volume, such as cube boundaries. Equivalently, almost every point must lie in exactly one translated copy. Periodicity is defined using three linearly independent real period vectors.

> **Theorem.** There is a finite nonempty set $F \subseteq \mathbb{Z}^3$ that tiles $\mathbb{Z}^3$ by translations but admits no periodic translational tiling. Moreover,
>
> $$K = \bigcup_{f \in F} \left(f + [0,1]^3 \right)$$
>
> tiles $\mathbb{R}^3$ by translations, up to sets of volume zero, but admits no periodic translational tiling, even when arbitrary real translation vectors are allowed.

An integer-lattice tiling of F immediately gives a Euclidean tiling of K by filling in the unit cubes. The stronger part is that choosing noninteger placements of K cannot produce a periodic arrangement.

Together with the positive results in dimensions one and two, the discrete conclusion shows that dimension three is the first dimension in which a finite integer-lattice tile can force the failure of full periodicity.

How can one finite tile enforce such a global restriction?

The proof starts with an arithmetic array resembling an infinite Sudoku puzzle. Its entries come from a finite alphabet, and certain words sampled along lines must satisfy prescribed arithmetic conditions.

Arrays satisfying the conditions exist, but none of the required arrays can repeat vertically. A hypothetical positive period leads, after a rescaling, to a smaller positive period obtained by dividing by a fixed prime. Repeating the argument gives a contradiction.

The proof then encodes these constraints into simultaneous tiling equations and combines them into a single tile.

The crucial step for obtaining dimension three is to keep the additional finite information in one cyclic coordinate: an integer read modulo a fixed positive integer Q. This coordinate can be represented using the third integer coordinate alongside the two coordinates carrying the arithmetic construction.

Every hypothetical fully periodic tiling can be decoded in the reverse direction. It would produce an arithmetic array with the forbidden vertical period.

Finally, a geometric rigidity construction handles the passage to Euclidean space. It ensures that a periodic tiling using arbitrary real translations would still yield a forbidden periodic discrete structure.

The theorem locates the first failure of the periodic tiling conjecture for integer lattices exactly at dimension three. A single finite pattern can fill space while preventing any finite patch from serving as a repeating cell.

References

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