The Triangular Lattice Is Universally Optimal

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In a preprint dated September 23, 2026, OpenAI (OpenAI 2026) proved that the triangular lattice is universally optimal in the plane. The same arrangement minimizes energy for an entire class of repulsive interactions, among all point configurations of the same density.

In my earlier post on sphere packing, we considered how densely nonoverlapping balls can fill space. Energy minimization allows interactions between particles at every distance. Changing the interaction can change the best arrangement, which makes universal optimality a particularly strong property.

Let us explain the definitions.

The *triangular lattice* consists of points arranged in staggered rows, with six nearest neighbours around each point. Joining nearest neighbours produces a tiling by equilateral triangles.

We normalize its density to one:

$$\Lambda_\triangle = \sqrt{\frac{2}{\sqrt{3}}} \left\{ \left(m + \frac{n}{2}, \frac{\sqrt{3}n}{2} \right) : m, n \in \mathbb{Z} \right\}.$$

The two generating vectors span a parallelogram of area one, so there is one lattice point per unit area on average.

To compare this lattice with other arrangements, we must fix the density.

A point configuration $C \subseteq \mathbb{R}^2$ is *locally finite* if every bounded disk contains only finitely many points. Let B_R be the closed disk of radius R centred at the origin, and write

$$C_R = C \cap B_R, \qquad N_R = |C_R|.$$

We say that C has density one if

$$\frac{N_R}{\pi R^2} \longrightarrow 1 \qquad (R \longrightarrow \infty).$$

Suppose that two particles at positions x and y contribute

$$g(|x-y|^2)$$

to the interaction energy, where $|x-y|$ is their Euclidean distance.

The *lower energy per particle* is

$$E_g(C) = \liminf_{R\to\infty} \frac{1}{N_R} \sum_{\substack{x,y\in C_R \\ x\neq y}} g(|x-y|^2).$$

The sum counts ordered pairs. Using unordered pairs would divide all energies by two and give the same minimizers.

The lower limit is used because an irregular configuration need not have a limiting average. The value $+\infty$ is allowed.

Which interactions are covered by universal optimality?

A smooth function $g : (0,\infty) \to \mathbb{R}$ is *completely monotone* if

$$(-1)^k g^{(k)}(t) \geq 0 \qquad (k = 0, 1, 2, \ldots;\ t > 0).$$

Thus g is nonnegative and decreasing, its second derivative is nonnegative, and the signs of its higher derivatives continue to alternate.

Important examples are

$$g(t) = \exp(-\alpha t), \qquad \alpha > 0,$$

and

$$g(t) = t^{-s/2}, \qquad s > 0.$$

As functions of the distance r, these give the Gaussian interaction $\exp(-\alpha r^2)$ and the inverse-power interaction r^{-s}.

A configuration is *universally optimal* if it minimizes energy for every completely monotone function of squared distance, among all configurations of the same density.

The triangular lattice had long been known to have remarkable energy-minimizing properties. In 1988, Montgomery (Montgomery 1988) proved that it minimizes every Gaussian interaction among planar lattices of a fixed density.

The restriction to lattice competitors was substantial. A general competitor can have several points in a repeating cell, or have no repeating pattern at all.

Universal optimality among all configurations was subsequently proved in dimensions eight and twenty-four for two special lattices, called the E_8 and Leech lattices, by Cohn, Kumar, Miller, Radchenko and Viazovska (Cohn et al. 2022). The planar problem remained open.

The new result is the following.

> **Theorem.** The triangular lattice is universally optimal in $\mathbb{R}^2$.

More explicitly, for every completely monotone g and every locally finite planar configuration C of density one,

$$E_g(C) \geq E_g(\Lambda_\triangle) = \sum_{v \in \Lambda_\triangle \setminus \{0\}} g(|v|^2).$$

The lattice sum gives the interaction of one lattice point with all the others.

Every locally finite competitor of the specified density is allowed, including nonperiodic configurations with very uneven local arrangements. Other positive densities follow by rescaling.

For Gaussian interactions, the lattice energy is finite. For inverse powers r^{-s} in the plane, it is finite when $s > 2$; the theorem also permits the infinite energies arising for longer-range interactions.

The proof begins with Gaussians.

Completely monotone functions can be represented as nonnegative weighted mixtures of exponential functions. For squared-distance interactions, these become mixtures of Gaussians. The main analytic task is therefore to establish a sharp energy bound for every Gaussian parameter.

The method uses auxiliary functions and their Fourier transforms. For a sufficiently rapidly decaying function f on $\mathbb{R}^2$, the Fourier transform is

$$\widehat{f}(\xi) = \int_{\mathbb{R}^2} f(x) \exp(-2\pi i\, x \cdot \xi)\, dx.$$

For each $\alpha > 0$, the proof constructs a smooth, rapidly decaying real function f_α, depending only on $|x|$, such that

$$f_\alpha(x) \leq \exp(-\alpha |x|^2), \qquad \widehat{f_\alpha}(\xi) \geq 0.$$

These inequalities yield the lower bound

$$E_{g_\alpha}(C) \geq \widehat{f_\alpha}(0) - f_\alpha(0), \qquad g_\alpha(t) = \exp(-\alpha t),$$

for every admissible configuration C.

The functions are constructed so that this lower bound equals the triangular-lattice energy exactly. Achieving the required contacts with the Gaussian while keeping the Fourier transform nonnegative is the central difficulty.

Once the sharp Gaussian comparisons are established, positive mixtures and limiting arguments give the full class of completely monotone interactions.

The theorem reveals a remarkable robustness of the triangular pattern. Its optimality persists as the interaction changes its range and shape throughout this class, and as competitors range over arbitrary arrangements of the same density. The familiar triangular grid is an energy minimizer for infinitely many different problems at once.

References

Cohn, Henry, Abhinav Kumar, Stephen D. Miller, Danylo Radchenko, and Maryna Viazovska. 2022. “Universal Optimality of the E_8 and Leech Lattices and Interpolation Formulas.” *Annals of Mathematics* 196 (3): 983–1082. https://doi.org/10.4007/annals.2022.196.3.3.

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