# On Kaplansky’s idempotent conjecture

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Two OpenAI preprints, dated September 24 and October 4, 2026, give contrasting answers to longstanding questions about group rings (OpenAI 2026b, 2026a). Nontrivial idempotents can occur over the two-element field even for torsion-free groups, while characteristic zero excludes them.

Let us explain these statements and their connection.

Let $G$ be a group, and let $R$ be a commutative ring with identity. The *group ring* $R[G]$ consists of formal sums

$$u = \sum_{g \in G} r_g g, \qquad r_g \in R,$$

in which only finitely many coefficients are nonzero.

Addition is performed coefficient by coefficient. Multiplication combines the multiplication in $R$ with the group operation:

$$\left(\sum_g r_g g\right)\left(\sum_h s_h h\right) = \sum_{g,h} r_g s_h (gh).$$

Terms representing the same group element are collected together. The identity is the formal sum supported only at the group identity $1_G$, with coefficient $1_R$; we write it simply as $1$.

Although the coefficient ring is commutative, the group ring need not be, because $gh$ and $hg$ may differ.

For example, if $G$ is the infinite cyclic group generated by $t$, its group ring consists of finite expressions

$$r_m t^m + \cdots + r_n t^n,$$

where negative exponents are allowed. These are Laurent polynomials.

A group is *torsion-free* if

$$g^n = 1_G, \qquad n \geq 1,$$

implies $g = 1_G$. Thus every nonidentity element has infinite order.

An element $p$ of a ring is called an *idempotent* if

$$p^2 = p.$$

The elements $0$ and $1$ always satisfy this identity and are called the trivial idempotents.

Finite-order group elements give an elementary source of nontrivial idempotents. If $g$ has order $n \geq 2$, then in $\mathbb{Q}[G]$ the element

$$p = \frac{1}{n}(1 + g + \cdots + g^{n-1})$$

satisfies $p^2 = p$. Indeed, each power of $g$ occurs exactly $n$ times when the square is expanded. The resulting idempotent is neither $0$ nor $1$.

This explains the torsion-free hypothesis in *Kaplansky's idempotent conjecture*: for a torsion-free group $G$ and a field $K$, the conjecture predicted that $K[G]$ has only the trivial idempotents. The problem appears in Kaplansky's book (Kaplansky 1972, pt. II, Section 3).

A second question concerns inverses.

For square matrices over a field, $AB = I$ implies $BA = I$. A ring with the analogous property

$$ab = 1 \quad \Longrightarrow \quad ba = 1$$

is called *directly finite*. Kaplansky's direct-finiteness conjecture predicted this property for $K[G]$ for every field $K$ and every group $G$.

The October preprint disproves it even for a torsion-free group.

Write $\mathbb{F}_2 = \{0, 1\}$ for the two-element field, whose arithmetic satisfies $1 + 1 = 0$. Recall also that a group is *finitely presented* if finitely many generators and finitely many defining relations suffice to describe it.

> **Theorem (characteristic two).** There exist a finitely presented torsion-free group $G$ and elements $a, b \in \mathbb{F}_2[G]$ such that
>
> $$ab = 1, \qquad ba \neq 1.$$

The connection with idempotents is immediate once we set

$$p = 1 - ba.$$

Since

$$(ba)^2 = b(ab)a = ba,$$

we obtain

$$p^2 = 1 - 2ba + (ba)^2 = 1 - ba = p.$$

Moreover, $p \neq 0$ because $ba \neq 1$.

Also $p \neq 1$. Otherwise $ba = 0$, which would imply

$$b = b(ab) = (ba)b = 0,$$

contradicting $ab = 1$.

Thus the same example disproves the idempotent conjecture in characteristic two.

It also gives nonzero zero divisors. A nonzero element $u$ is a *zero divisor* if $uv = 0$ for some nonzero $v$. Here

$$p(1 - p) = 0,$$

and both factors are nonzero. Consequently, the theorem also disproves Kaplansky's zero-divisor conjecture in characteristic two.

The characteristic-zero result goes in the other direction.

A commutative ring $R$ with identity is an *integral domain* if $1 \neq 0$ and a product of two nonzero elements is always nonzero. It has *characteristic zero* if

$$n \cdot 1_R \neq 0$$

for every positive integer $n$. Examples include $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$.

The September preprint proves the following.

> **Theorem (characteristic zero).** Let $G$ be a torsion-free group, and let $R$ be a commutative integral domain of characteristic zero. Every element $p \in R[G]$ satisfying $p^2 = p$ is either $0$ or $1$.

This proves the characteristic-zero idempotent conjecture, with coefficient domains allowed as well as fields.

In particular, $R[G]$ cannot be a product of two nonzero unital rings: such a product would contain the nontrivial idempotent $(1, 0)$.

The proofs use quite different mechanisms.

The characteristic-two construction uses finite labeled graphs with prescribed parity properties. Terms in products of group-ring elements are arranged to cancel in pairs, using $1 + 1 = 0$, while one product retains exactly the identity term. Geometric arguments ensure that the resulting group is torsion-free and that the algebraic witness to the failed reverse identity remains nonzero.

For characteristic zero, the main input is a stronger result about traces, resolving the complex group-ring Bass trace conjecture.

For a complex idempotent

$$p = \sum_g c_g g,$$

consider the two quantities

$$\varepsilon(p) = \sum_g c_g, \qquad \tau(p) = c_{1_G}.$$

The first is called the *augmentation*; the second extracts the coefficient of the identity.

Augmentation respects multiplication, so $p^2 = p$ implies

$$\varepsilon(p)^2 = \varepsilon(p), \qquad \varepsilon(p) \in \{0, 1\}.$$

For torsion-free groups, the new trace theorem gives

$$\tau(p) = \varepsilon(p).$$

A classical operator-algebra argument then forces $p$ itself to be $0$ or $1$. A reduction involving its finitely many coefficients extends the conclusion to every characteristic-zero integral domain.

The coefficient characteristic therefore makes a decisive difference. Torsion-free groups exclude nontrivial idempotents in characteristic zero, while characteristic two permits them even for a finitely presented group, together with a one-sided inverse that fails to be two-sided.

## References


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