Howie’s Conjecture on Group Equations

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In a preprint dated October 5, 2026, OpenAI (OpenAI 2026) proved Howie’s conjecture on systems of equations over groups. A linear independence condition guarantees that the equations can be solved in a larger group while preserving the original group.

The possibility of enlarging the group is essential. For example, in the additive group of integers, the equation

$$2x = 1$$

has no solution. It does have a solution in the additive group of rational numbers, which contains the integers.

The new theorem establishes a much broader principle for groups whose multiplication need not be commutative.

Write the group operation multiplicatively, with identity element 1. An *equation over a group* G is an expression

$$w(x_1, \dots, x_n) = 1,$$

where w is a finite product of unknowns, their inverses, and specified elements of G, called *coefficients*.

For example,

$$xayxb^{-1} = 1$$

is an equation in the unknowns x, y, with coefficients $a, b \in G$. The order of the factors matters.

An *overgroup* of G is a group H containing an isomorphic copy of G. Thus the multiplication of the original elements is preserved, and distinct elements of G remain distinct.

A simultaneous solution of a system in an overgroup consists of one such group H and elements of H that satisfy all the equations at once.

Some equations cannot be solved even after enlarging the group. If $a \in G$ is not the identity, then

$$xax^{-1} = 1$$

has no solution in any overgroup: multiplying by x^{-1} on the left and x on the right would give $a = 1$.

We therefore need a condition on the equations.

Consider a finite system

$$w_1(x_1, \dots, x_n) = 1, \quad \dots, \quad w_m(x_1, \dots, x_n) = 1.$$

For each equation w_i, count the total exponent of each unknown x_j. An occurrence of x_j contributes 1, and an occurrence of x_j^{-1} contributes -1. Coefficients contribute nothing.

These counts form an integer matrix

$$A = (a_{ij}),$$

with one row for each equation and one column for each unknown.

The system is called *nonsingular* if the rows of A are linearly independent over $\mathbb{Q}$. Equivalently, the row rank is m. In particular, there must be at least as many unknowns as equations.

For a square matrix, this condition is simply

$$\det A \neq 0.$$

For example, consider

$$xayxb^{-1} = 1, \qquad ycx^{-1}yd^{-1} = 1,$$

where a, b, c, d are arbitrary elements of G. Their exponent-sum matrix is

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}, \qquad \det A = 5.$$

The system is therefore nonsingular, regardless of the coefficients.

The matrix records only the total exponents. The full equations retain the order of the factors and all interactions with the coefficients. The remarkable assertion is that this limited linear information already guarantees solvability in an overgroup.

Gerstenhaber and Rothaus (Gerstenhaber and Rothaus 1962) proved the finite-group case in 1962. In 1981, Howie (Howie 1981) proposed the general nonsingular-system conjecture. The new theorem settles it for arbitrary coefficient groups.

> **Theorem.** Every finite nonsingular system of equations over an arbitrary group has a simultaneous solution in an overgroup.

In particular, the two equations above can always be solved simultaneously in some group containing G, for every choice of a, b, c, d.

For one equation in one unknown, nonsingularity means that the total exponent of the unknown is nonzero. Thus, for every $g \in G$ and every nonzero integer r, the equation

$$x^r = g$$

has a solution in an overgroup. The theorem also applies when occurrences of the unknown are separated by coefficients, so that the equation cannot be reduced to a root equation.

The result gives a precise condition under which equations can be imposed without collapsing the group supplying their coefficients. A test involving only an integer matrix controls solvability over every group, including groups with highly complicated multiplication.

References

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