Gersten’s Hyperbolicity Conjecture

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In a preprint dated September 25, 2026, OpenAI (OpenAI 2026) proved Gersten’s conjecture on one-relator groups. The result characterizes their hyperbolicity by excluding an explicit family of subgroups.

Let us explain the statement.

A group is *finitely generated* if there are finitely many elements $x_1, \dots, x_d$ such that every element is a product of these generators and their inverses. A *word* is a finite sequence of generators and inverse generators.

A *one-relator group* has a presentation

$$G = \langle x_1, \dots, x_d \mid r = 1 \rangle,$$

where r is one word and 1 denotes the identity. This means that all equalities in the group follow from the group axioms and the single defining relation $r = 1$.

For example,

$$\langle a, b \mid aba^{-1}b^{-1} = 1 \rangle = \langle a, b \mid ab = ba \rangle \cong \mathbb{Z}^2.$$

The relation allows the generators to commute, and the resulting group is the group of pairs of integers under coordinatewise addition.

One-relator groups also appeared in my earlier post on their infinite-index subgroups. Here the question concerns their geometry.

Choose a finite generating set S containing the inverse of each of its elements. The *Cayley graph* of G has one vertex for each group element, with an edge of length one from g to gs for every $s \in S$. Distances are measured by shortest paths, including points inside edges.

A *geodesic* is a shortest path between its endpoints. Three points, together with a choice of geodesics joining them, form a geodesic triangle.

The triangle is δ*-thin* if every point on each side lies within distance δ of the union of the other two sides. A group is *word-hyperbolic*, or simply *hyperbolic*, if there is a constant $\delta \geq 0$ such that every geodesic triangle in its Cayley graph is δ-thin.

The same constant must work for triangles of every size. This property is independent of the finite generating set, although δ may change.

This is the notion introduced in Gromov's theory (Gromov 1987) and discussed in my recent post on hyperbolic groups. Trees provide the simplest examples: their geodesic triangles collapse to tripods and are 0-thin.

The square grid behaves differently. In the usual Cayley graph of $\mathbb{Z}^2$, take vertices

$$(0,0), \qquad (N,0), \qquad (0,N).$$

Choose the two coordinate-axis segments as two sides, and let the third side travel through (N,N). All three sides are geodesics. The point (N,N) is at distance N from the union of the other two sides. Since N can be arbitrarily large, no uniform thinness constant exists.

Thus a single defining relation can already produce a nonhyperbolic group.

A broader family of examples is given by the *Baumslag–Solitar groups*

$$\mathrm{BS}(m,n) = \langle a, t \mid ta^m t^{-1} = a^n \rangle, \qquad m,n \in \mathbb{Z} \setminus \{0\}.$$

The parameters may have either sign.

The simplest case is

$$\mathrm{BS}(1,1) \cong \mathbb{Z}^2,$$

because its defining relation says that a and t commute.

Another instructive example is

$$\mathrm{BS}(1,2) = \langle a, t \mid tat^{-1} = a^2 \rangle.$$

Repeated conjugation gives

$$t^k a t^{-k} = a^{2^k} \qquad (k \geq 0).$$

Consequently, the power a^{2^k} can be represented by a word of length $2k+1$.

Here a has infinite order. One way to see this is to consider the transformations of the real line

$$a(x) = x + 1, \qquad t(x) = 2x.$$

They satisfy $tat^{-1} = a^2$, and the translation a has infinite order.

Such exponential shortcuts along the powers of an infinite-order element cannot occur in a hyperbolic group: the word lengths of those powers grow at least linearly with the exponent.

More generally, no hyperbolic group contains a subgroup isomorphic to any $\mathrm{BS}(m,n)$ with $m,n \neq 0$. These groups are therefore unavoidable subgroup obstructions to hyperbolicity.

In 1992, Gersten (Gersten 1992, 228) asked whether they are the only such obstructions for one-relator groups. The new theorem answers this affirmatively.

> **Theorem.** Let G be a finitely generated one-relator group. If G contains no subgroup isomorphic to $\mathrm{BS}(m,n)$ for any nonzero integers m, n, then G is word-hyperbolic.

Together with the known obstruction, this gives an equivalence: a finitely generated one-relator group is hyperbolic precisely when it contains no Baumslag–Solitar subgroup.

The hypothesis concerns subgroups anywhere inside G. Their generators may be complicated words in the original generators, so the obstruction need not be apparent from the displayed defining relation.

The proof proceeds through decompositions into smaller one-relator pieces. Induction provides hyperbolic geometry for the smaller groups, but controlling how that geometry behaves when the pieces are joined requires a substantial combination argument.

A central step establishes uniform expansion for sufficiently wide strips of paths crossing successive gluings. Failure of the required expansion would force a Baumslag–Solitar subgroup. Excluding these subgroups therefore supplies the geometric control needed to preserve hyperbolicity. Much of the work handles the complicated intersections among the subgroups used in the decompositions.

The theorem identifies a precise connection between subgroup structure and global geometry. Within the one-relator class, every failure of uniformly thin triangles is witnessed by a subgroup from the explicit Baumslag–Solitar family.

References

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