A Group Without Fixed Price

Bogdan Grechuk · 25 Nov 2026

https://functor.network/user/3333/entry/2098

In a preprint dated October 5, 2026, OpenAI (OpenAI 2026) constructed a finitely generated group with two free measure-preserving actions of different costs. This answers Gaboriau's fixed-price question negatively.

The word "cost" has a precise mathematical meaning here. It measures how economically the orbits of an action can be connected by measurable partial transformations. The notion was introduced by Levitt (Levitt 1995) and developed by Gaboriau (Gaboriau 2000).

Let us explain the definitions.

An *action* of a group G on a set X assigns to each $g \in G$ a bijection $x \mapsto g \cdot x$, with

$$e \cdot x = x, \qquad (gh) \cdot x = g \cdot (h \cdot x),$$

where e is the identity of G. The *orbit* of x is

$$G \cdot x = \{g \cdot x : g \in G\}.$$

The orbits partition X.

Suppose that G is countable and that X carries a probability measure μ, so that $\mu(X) = 1$. We use standard probability spaces: up to sets of measure zero, these can be represented by an interval with length measure together with at most countably many points of positive measure.

The action is *measure-preserving* if its transformations are measurable and

$$\mu(g \cdot A) = \mu(A)$$

for every measurable set A and every $g \in G$. It is *essentially free* if, for every $g \neq e$,

$$\mu(\{x \in X : g \cdot x = x\}) = 0.$$

Thus, outside a set of measure zero, distinct group elements send any given point to distinct points.

To define cost, choose a countable family of partial transformations

$$\varphi_i : A_i \longrightarrow g_i \cdot A_i, \qquad \varphi_i(x) = g_i \cdot x,$$

where each A_i is measurable and $g_i \in G$.

Draw an undirected edge between x and $\varphi_i(x)$ whenever $x \in A_i$. The family is called a *generating graphing* if, after discarding an invariant set of measure zero, the connected components of this graph are exactly the orbits of the action. In other words, these partial moves and their inverses allow us to travel between any two points in the same orbit.

The cost of the graphing $\Phi = (\varphi_i)$ is

$$\mathcal{C}(\Phi) = \sum_i \mu(A_i).$$

A transformation available everywhere contributes 1, while one available only on a set of measure p contributes p.

The *cost of the action* is

$$\mathcal{C}(G \curvearrowright X) = \inf\{\mathcal{C}(\Phi) : \Phi \text{ is a generating graphing}\}.$$

The infimum allows increasingly efficient graphings; a graphing attaining it need not exist.

There is a simple finite analogy. Connecting N vertices requires at least $N - 1$ edges, giving a minimum of $1 - 1/N$ edges per vertex. Cost replaces normalized counting by probability measure and requires the connections to be chosen measurably.

If G is generated by d elements, using each generator on all of X gives a generating graphing of cost d. Partial transformations may allow a substantial saving.

A group has *fixed price* if all its essentially free measure-preserving actions on standard probability spaces have the same cost.

For example, the integers $\mathbb{Z}$ have fixed price 1. More generally, the free group F_n, whose elements are words in n generators and their inverses with only inverse-pair cancellations imposed, has fixed price n (Gaboriau 2000). Although the generators immediately give the upper bound n, proving that no generating graphing can do better is a substantial theorem.

Gaboriau also proved that Thompson’s group F, discussed in my recent post, has fixed price 1.

In 2000, Gaboriau (Gaboriau 2000) asked whether every countable group has fixed price. The new result gives a counterexample even among finitely generated groups.

> **Theorem.** There exist a finitely generated group G and two essentially free measure-preserving actions on standard probability spaces (X, μ) and (Y, ν) such that
>
> $$\mathcal{C}(G \curvearrowright X) \neq \mathcal{C}(G \curvearrowright Y).$$

The construction gives a more precise comparison. There is a fixed constant $\eta > 0$ and, for every integer $M > 100$, an action on a space Y_M such that

$$\mathcal{C}(G \curvearrowright X) \geq 1 + \eta, \qquad 1 \leq \mathcal{C}(G \curvearrowright Y_M) \leq 1 + \frac{99}{M}.$$

For sufficiently large M, the second cost is strictly smaller than the first.

The *cost of a group* is the infimum of the costs of all its essentially free measure-preserving actions. Every such action of an infinite group has cost at least 1. Consequently, the constructed group has group cost 1, while its action on X has cost strictly greater than 1. Group cost 1 and fixed price 1 are therefore different properties.

The two actions have a concrete probabilistic description.

For the first, independently assign a uniformly distributed number in $[0, 1]$ to every element of G. The space of all assignments is

$$X = [0, 1]^G,$$

with its product probability measure. The group acts by reindexing the assignments. This is the *Bernoulli action*, and it is essentially free.

For the second action, add a uniformly distributed coordinate taking M possible values. Certain group elements shift this coordinate cyclically. It allows some partial transformations to be retained on only a small fraction of the coordinate levels; relations in the group recover the missing connections. This produces the upper bound $1 + 99/M$.

The lower bound for the Bernoulli action is harder. The proof passes from measurable graphings to finite permutation models and shows that the resulting finite groups require many generators. These estimates force a positive extra cost η.

Every orbit of a free action can be identified with the group itself. The difference revealed by the theorem comes from arranging connections measurably across all the orbits at once. Their most efficient measurable connection can depend on the action, even when the group remains the same.

References

Gaboriau, Damien. 2000. “Coût Des Relations d’équivalence Et Des Groupes.” *Inventiones Mathematicae* 139 (1): 41–98. https://doi.org/10.1007/s002229900019.

Levitt, Gilbert. 1995. “On the Cost of Generating an Equivalence Relation.” *Ergodic Theory and Dynamical Systems* 15 (6): 1173–81. https://doi.org/10.1017/S0143385700009846.

OpenAI. 2026. *A Group Without Fixed Price*. Preprint. https://github.com/openai/math/blob/main/preprints/A-group-without-fixed-price-October-5-2026/unequal-costs-rank-100-amalgam.pdf.