Hyperbolic but Not Residually Finite Groups

Bogdan Grechuk · 24 Nov 2026

https://functor.network/user/3333/entry/2095

In a preprint dated September 23, 2026, OpenAI (OpenAI 2026) constructed a torsion-free word-hyperbolic group that is not residually finite. This answers a longstanding question about the relationship between the geometry of an infinite group and its finite images.

Let us explain these conditions.

A homomorphism between groups is a map that respects multiplication:

$$\varphi(xy) = \varphi(x)\varphi(y).$$

A group G, with identity element e, is called *residually finite* if, for every $g \neq e$, there is a finite group Q and a homomorphism

$$\varphi : G \longrightarrow Q$$

such that $\varphi(g) \neq e_Q$, where e_Q is the identity of Q. Thus every nonidentity element remains visible in some finite image. The finite group and the homomorphism may depend on the element.

For example, the integers under addition are residually finite. Given a nonzero integer n, choose an integer $m > |n|$. Reduction modulo m maps the integers to a finite cyclic group and sends n to a nonzero residue.

More generally, residual finiteness allows any two distinct elements $x, y \in G$ to be distinguished in a finite image: apply the definition to $x^{-1}y$.

This property also appeared in my earlier post on the Burnside problem. There, an infinite finitely presented group could still have all its individual elements detected in finite groups. The new theorem concerns a geometric condition that might have been expected to guarantee the same possibility.

To describe that geometry, suppose that G has a finite generating set S. Every element of G can then be expressed as a product of elements of S and their inverses. We may include these inverses in S, so that S is *symmetric*.

The *Cayley graph* of G has one vertex for each group element, with an edge from g to gs for each $s \in S$. Give every edge length one, and measure distances by the lengths of shortest paths. Points inside edges are included.

A *geodesic* is a shortest path between its endpoints. Three points, together with a choice of geodesics joining them, form a geodesic triangle.

Such a triangle is called δ*-thin* if every point on each side lies within distance δ of the union of the other two sides. The group is *word-hyperbolic*, or simply *hyperbolic*, if there is one constant $\delta \geq 0$ for which every geodesic triangle in its Cayley graph is δ-thin.

The same constant must work for triangles of every size. This condition is independent of the chosen finite generating set, although the value of δ may change.

For an elementary example, the Cayley graph of the integers with generators 1 and -1 is a line. More generally, triangles in a tree collapse to tripods: each side is contained in the union of the other two. They are therefore 0-thin. Hyperbolicity permits a uniformly bounded departure from this behavior.

Gromov’s 1987 work (Gromov 1987) established hyperbolic groups as a central subject in geometric group theory. A major question was whether every hyperbolic group is residually finite.

The remaining condition in the new result is *torsion-freeness*. A group is torsion-free if

$$g^n = e, \qquad n \geq 1,$$

implies $g = e$. Equivalently, every nonidentity element has infinite order.

The new theorem is the following.

> **Theorem.** There exists a torsion-free word-hyperbolic group that is not residually finite.

For this group, there is a fixed element $g \neq e$ such that

$$\varphi(g) = e_Q$$

for every finite group Q and every homomorphism $\varphi : G \to Q$. However large a finite image we choose, this same nonidentity element disappears.

The theorem also has a consequence through an earlier result of Kapovich and Wise (Kapovich and Wise 2000). In 2000, they proved that the following two assertions are equivalent: every hyperbolic group is residually finite; and every

nontrivial hyperbolic group has a nontrivial finite quotient. Consequently, the new theorem implies that some nontrivial hyperbolic group has no nontrivial finite quotient at all. This further conclusion concerns the class as a whole; the group supplied by the equivalence need not be the particular torsion-free example above.

The construction combines geometry with linear algebra.

The group is the fundamental group of a finite space assembled from Euclidean triangles. Its elements are represented by closed paths based at a chosen point, with paths identified when one can be continuously deformed into the other while keeping the base point fixed. Multiplication joins paths end to end.

Control of the angles at the vertices establishes hyperbolicity and torsion-freeness. It also ensures that a particular finite collection of short loops represents nonidentity elements.

Suppose that every element in this collection survived in some finite image. Taking the product of these finitely many images would give one finite image in which they all survived simultaneously.

The proof encodes that image using permutation matrices. The surviving elements force an associated matrix to have large rank. Polynomial interpolation over a finite field gives an incompatible upper bound on the same rank. The contradiction shows that at least one of the chosen nonidentity elements vanishes in every finite image.

The result establishes a limit to what thin triangles can guarantee. Even a torsion-free group with this strong geometric regularity can contain elements that no finite group can distinguish from the identity. Hyperbolicity alone therefore does not ensure residual finiteness.

References

Gromov, Mikhail. 1987. “Hyperbolic Groups.” In *Essays in Group Theory*, edited by S. M. Gersten, vol. 8. Mathematical Sciences Research Institute Publications. Springer. https://doi.org/10.1007/978-1-4613-9586-7_3.

Kapovich, Ilya, and Daniel T. Wise. 2000. “The Equivalence of Some Residual Properties of Word-Hyperbolic Groups.” *Journal of Algebra* 223 (2): 562–83. https://doi.org/10.1006/jabr.1999.8104.

OpenAI. 2026. *A Torsion-Free Hyperbolic Group That Is Not Residually Finite*. Preprint. https://github.com/openai/math/blob/main/preprints/a-torsion-free-hyperbolic-group-that-is-not-residually-finite-September-23-2026/paper.pdf.