Simple, Amenable, Finitely Presented Groups

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https://functor.network/user/3333/entry/2094

In a preprint dated September 23, 2026, OpenAI (OpenAI 2026) constructed an infinite finitely presented simple amenable group. The statement brings together four conditions whose compatibility had remained a long-standing problem.

We recently discussed amenability in my post on Thompson's group. Let us recall the definitions needed to understand the new theorem.

A subgroup N of a group G is called *normal* if

$$gNg^{-1} = N \qquad \text{for every } g \in G.$$

A nontrivial group is *simple* if its only normal subgroups are $\{e\}$ and G, where e is the identity element.

Normal subgroups allow us to form quotient groups, so simplicity means that the group has no nontrivial proper quotient. The word "simple" describes this structural property; it does not mean that the group is easy to understand.

A group is *finitely generated* if every element can be expressed using finitely many specified generators and their inverses. It is *finitely presented* if finitely many defining relations also suffice. In notation,

$$G = \langle s_1, \dots, s_r \mid w_1 = e, \dots, w_m = e \rangle,$$

where each w_j is a word in the generators and their inverses, and every relation in G follows from the displayed relations and the group axioms.

Finite presentation is therefore stronger than finite generation. A finite collection of generators may require infinitely many defining relations.

Amenability can be expressed by the Følner criterion (Følner 1955). A group G is *amenable* if, for every finite subset $K \subseteq G$ and every $\epsilon > 0$, there is a nonempty finite subset $D \subseteq G$ such that

$$|gD \triangle D| < \epsilon |D| \qquad (g \in K).$$

Here $gD = \{gd : d \in D\}$, and $\triangle$ denotes symmetric difference. Thus finite sets can be made almost unchanged by any prescribed finite collection of translations.

Some familiar groups satisfy several of these conditions. A cyclic group of prime order is simple, amenable, and finitely presented, but finite. The integers under addition are infinite, amenable, and finitely presented, but not simple: the even integers form a proper nontrivial normal subgroup.

The new theorem achieves all four conditions simultaneously.

Theorem 1 *There exists an infinite finitely presented simple amenable group.*

The major earlier breakthrough came in 2013, when Juschenko and Monod (Juschenko and Monod 2013) established the existence of infinite finitely generated simple amenable groups. Their examples arose from transformations of Cantor spaces, combining an amenability theorem with earlier structural results.

However, these groups were not finitely presented. On page 776, Juschenko and Monod explicitly wrote: “The next problem would be to find finitely presented examples.” The new theorem answers this question affirmatively.

An interesting consequence of simplicity is that the new group has no nontrivial finite images. Indeed, let

$$\phi : G \longrightarrow H$$

be a homomorphism to a finite group, meaning that $\phi(xy) = \phi(x)\phi(y)$. Its kernel

$$\ker \phi = \{g \in G : \phi(g) = e_H\}$$

is normal. Simplicity forces this kernel to be either $\{e\}$ or G. In the first case, ϕ would be injective, which is impossible because G is infinite and H is finite. Hence the kernel is G, and ϕ is trivial.

Compare this with the residually finite group in my recent post on the Burnside problem, where finite images detect every nonidentity element.

This also clarifies the meaning of amenability. The finite sets in the Følner criterion are subsets of the group. They need not be finite subgroups or images under homomorphisms. Almost-invariant finite sets can exist even when every finite image is trivial.

The construction has a geometric description. It uses finitely many square boards cut into polygonal pieces, with arithmetic restrictions involving the golden ratio

$$\tau = \frac{1+\sqrt{5}}{2}.$$

The group is built from local moves that cyclically permute three disjoint pieces which are translates of one another. Such three-cycles generate the local even permutations used in the construction.

The simplicity argument shows that every nontrivial normal subgroup must contain enough of these local moves to generate the whole group.

Amenability requires a separate argument. The proof constructs random polygonal partitions whose distributions are almost unchanged by the required translations. Matching cells of the same shape then produces almost-invariant measures on the exchange group, and ultimately finite sets satisfying the Følner criterion.

Finite presentation is the most delicate additional step. The proof constructs a finitely presented group mapping onto the desired group. Its kernel is central, meaning that its elements commute with every element of the covering group. A further argument shows that this kernel is finitely generated. Adding finitely many relations that set its generators equal to the identity then gives a finite presentation of the desired group.

The theorem shows that invariant averaging, the absence of proper nontrivial normal subgroups, and a finite list of defining relations can coexist in one infinite group.

References

Følner, Erling. 1955. “On Groups with Full Banach Mean Value.” *Mathematica Scandinavica* 3: 243–54. https://doi.org/10.7146/math.scand.a-10442.

Juschenko, Kate, and Nicolas Monod. 2013. “Cantor Systems, Piecewise Translations and Simple Amenable Groups.” *Annals of Mathematics* 178 (2): 775–87. https://doi.org/10.4007/annals.2013.178.2.7.

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