Thompson’s Group F Is Nonamenable

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In a preprint dated September 23, 2026, OpenAI (OpenAI 2026) proved that Thompson’s group F is not amenable. This resolves a problem studied for decades and confirms a conjecture made by Ross Geoghegan in 1979, as recorded by Cannon, Floyd and Parry (Cannon et al. 1996, 227).

To understand the statement, we need to describe the group and explain amenability.

A *dyadic rational* is a number of the form $a/2^b$, where a is an integer and b is a nonnegative integer. Examples include $1/2$, $3/8$, and $5/16$.

Thompson’s group F consists of the continuous strictly increasing bijections

$$f : [0,1] \longrightarrow [0,1]$$

that are affine on each piece of a finite partition, with dyadic breakpoints and slopes that are integer powers of two. The allowed slopes include

$$\ldots, \quad \frac{1}{4}, \quad \frac{1}{2}, \quad 1, \quad 2, \quad 4, \quad \ldots.$$

The group operation is composition: $(fg)(t) = f(g(t))$.

For example, the map

$$f(t) = \begin{cases} t/2, & 0 \le t \le 1/2, \\ t - 1/4, & 1/2 \le t \le 3/4, \\ 2t - 1, & 3/4 \le t \le 1 \end{cases}$$

belongs to F. Its breakpoints are $1/2$ and $3/4$, and its slopes are $1/2$, 1, and 2.

Although its elements have arbitrarily complicated subdivisions, F is generated by just two transformations. It is even finitely presented: finitely many generators and defining relations suffice to describe the whole group (Cannon et al. 1996, sec. 3).

Amenability concerns the possibility of averaging in a way that is compatible with the group operation. A particularly concrete formulation is the Følner criterion (Følner 1955).

For a subset A of a group G and an element $s \in G$, write

$$sA = \{sa : a \in A\}.$$

Also, $A \triangle B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference.

A group G is *amenable* if, for every finite subset $S \subseteq G$ and every $\epsilon > 0$, there is a nonempty finite subset $A \subseteq G$ such that

$$|sA \triangle A| < \epsilon |A| \qquad \text{for every } s \in S.$$

Thus one can find finite sets that change by only a small fraction when translated by any prescribed finite collection of group elements.

Every finite group is amenable: take $A = G$. For an infinite example, consider the integers under addition. Put

$$A_N = \{-N, -N+1, \ldots, N\}.$$

For every fixed integer h,

$$\frac{|(h + A_N) \triangle A_N|}{|A_N|} \leq \frac{2|h|}{2N+1} \longrightarrow 0.$$

The same choice of sufficiently large N works simultaneously for any finite collection of shifts. Hence $\mathbb{Z}$ is amenable.

The new theorem says that Thompson's group behaves differently.

Theorem 1 *Thompson's group F is not amenable.*
Equivalently, there are a fixed nonempty finite set $S \subseteq F$ and a constant $\eta > 0$ such that every nonempty finite set $A \subseteq F$ satisfies

$$\max_{s \in S} |sA \triangle A| \geq \eta |A|.$$

Whatever finite collection of transformations we choose, composing its members with at least one map from S changes a fixed positive fraction of the collection. Here the sets being counted are collections of transformations of $[0, 1]$.

One reason the problem resisted the usual methods is a theorem of Brin and Squier (Brin and Squier 1985). In 1985, they proved that F contains no nonabelian free subgroup.

A free subgroup on two generators is generated by elements a, b whose only relations are those obtained by cancelling adjacent inverse pairs. Such a subgroup is a classical obstruction to amenability: every group containing one is nonamenable. Brin and Squier's theorem therefore ruled out this familiar route to proving nonamenability of F.

Together with finite presentability, the new result makes F a particularly concrete example of a finitely presented nonamenable group without nonabelian free subgroups.

The proof connects the interval transformations with a fixed-point phenomenon in infinite-dimensional geometry.

Let ℓ^2 be the space of real sequences $x = (x_1, x_2, \ldots)$ satisfying $\sum_{n=1}^{\infty} x_n^2 < \infty$, with norm

$$\|x\| = \left(\sum_{n=1}^{\infty} x_n^2 \right)^{1/2}.$$

Write $B = \{x \in \ell^2 : \|x\| \le 1\}$ for its closed unit ball.

Benyamini and Sternfeld (Benyamini and Sternfeld 1983) proved that there is a map $T : B \to B$ and positive constants L, δ such that

$$\|T(x) - T(y)\| \le L\|x - y\|, \qquad \|T(x) - x\| \ge \delta$$

for all $x, y \in B$. The first inequality says that T is Lipschitz continuous. The second says that every point is moved by at least the same positive distance. In particular, there cannot even be a sequence of approximate fixed points whose displacement tends to zero.

The proof uses this map to assign vectors to finite dyadic partitions of the interval. The assignment is recursive: colors of smaller pieces are averaged, and T is applied to that average.

Elements of F can transport suitably chosen dyadic subintervals onto one another by affine maps. If F were amenable, almost-invariant finite sets would allow averages of the corresponding colors to be compared. The resulting estimates would produce points of B whose displacement under T is arbitrarily small, contradicting the lower bound δ.

In October 2026, Kim and Lodha (Kim and Lodha 2026) gave a streamlined exposition of the argument. Their proof develops the same underlying idea through invariant averaging on finite subsets of the dyadic rationals.

The result reveals uniform expansion in a group built from elementary piecewise-linear transformations. Its source lies in the interaction between dyadic self-similarity and infinite-dimensional geometry, beyond the classical obstruction provided by free subgroups.

References

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