Finite Orders, Infinite Group

Bogdan Grechuk · 23 Nov 2026

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In two preprints dated September 23 and October 5, 2026, OpenAI (OpenAI 2026a, 2026b) constructed an infinite group with finitely many generators and defining relations, in which every element has finite order. This answers the finitely presented Burnside problem negatively. The October paper gives the stronger conclusion that all element orders are powers of two and that individual elements can be detected in finite groups.

Let us explain these conditions.

A group is a set with an associative multiplication, an identity element e, and an inverse for every element. Multiplication need not be commutative. The *order* of an element g is the smallest positive integer m such that

$$g^m = e,$$

if such an integer exists. Otherwise, g has infinite order. A group is called *periodic*, or a *torsion group*, if every element has finite order.

Every finite group is periodic. The converse fails: a group can have infinitely many elements even though repeatedly multiplying any one element eventually returns to the identity.

The interesting question arises when the group is *finitely generated*. This means that there are finitely many elements $g_1, \dots, g_r$ such that every group element is a product of these generators and their inverses. In 1902, Burnside (Burnside 1902) asked whether a finitely generated periodic group must be finite.

For abelian groups, whose multiplication is commutative, the answer is yes. If the generators have orders $m_1, \dots, m_r$, every element can be written as

$$g_1^{a_1} \cdots g_r^{a_r}, \qquad 0 \le a_i < m_i.$$

There are at most $m_1 \cdots m_r$ such expressions.

Without commutativity, the problem is much harder. Knowing the orders of the generators gives little control over products in which they alternate. The periodicity assumption requires finite order for every such product.

In 1964, Golod (Golod 1964), using the dimension estimates developed with Shafarevich (Golod and Shafarevich 1964), constructed infinite finitely generated periodic groups. These examples could also be chosen to be residually finite, a property we will define shortly. The additional requirement of a finite presentation remained open.

A group is *finitely presented* if it can be specified by finitely many generators and finitely many defining relations. A relation is an equality between products of generators and their inverses; all other relations must follow from the defining ones and the group axioms.

For example,

$$\langle a \mid a^4 = e \rangle$$

describes the cyclic group with elements e, a, a^2, a^3.

Finite generation and finite presentation are different requirements. One could try to force periodicity by listing all words in the generators and imposing a finite-order relation for each:

$$w_1^{n_1} = e, \qquad w_2^{n_2} = e, \qquad w_3^{n_3} = e, \qquad \ldots.$$

This would give infinitely many relations. A finite presentation requires the finite order of every word to follow from a single finite list.

There is one more condition in the new theorem. A group G is *residually finite* if, for every $g \neq e$, there is a finite group F and a homomorphism $\phi : G \to F$ such that $\phi(g) \neq e_F$. Here a homomorphism respects multiplication:

$$\phi(xy) = \phi(x)\phi(y).$$

Thus every nonidentity element remains visible in some finite image. The finite image may depend on the element.

The stronger October result is the following.

Theorem 1 *There exists an infinite finitely presented residually finite group G such that every element has finite order equal to a power of two.*

The identity has order $1 = 2^0$, and other element orders belong to the list

$$2, \quad 4, \quad 8, \quad 16, \quad \ldots.$$

An infinite group with this property is called a 2-group.

The orders in the constructed group must be unbounded. To see why, suppose that every element had order at most 2^K for some fixed integer K. Since the orders are powers of two, every element would satisfy the same identity

$$g^{2^K} = e.$$

Zel’manov’s solution of the restricted Burnside problem for 2-groups

(Zel’manov 1992) implies that a finitely generated residually finite group satisfying such an identity is finite. Consequently, arbitrarily large powers of two occur as orders in the new group.

This illustrates an important distinction between the two statements

$$\text{for every } g,\ \text{there exists } k \text{ such that } g^{2^k} = e$$

and

$$\text{there exists } K,\ \text{for every } g, \quad g^{2^K} = e.$$

The theorem realizes the first situation, with no uniform choice of K.

One ingredient behind the construction has an elementary algebraic explanation. Work with matrices over the two-element field $\mathbb{F}_2$, where $1 + 1 = 0$. A matrix A is *nilpotent* if $A^m = 0$ for some positive integer m. Since

$$(I + A)^2 = I + A^2,$$

repeated squaring gives

$$(I + A)^{2^k} = I + A^{2^k}.$$

For $2^k \geq m$, the right-hand side is I. Thus nilpotence produces an invertible matrix whose order is a power of two.

The papers develop a much more elaborate version of this mechanism inside an infinite algebra specified by finitely many relations. Associated groups model elementary matrix operations. Much of the work is needed to ensure that finite presentation, infinitude and periodicity hold simultaneously. The October paper then establishes separation by finite images and obtains the subgroup in which every element has power-of-two order.

The result combines several different kinds of finiteness: finitely many generators, finitely many defining relations, finite order for each element, and detection by finite groups. Even together, these conditions allow the group itself to be infinite.

References

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