Graham’s Rearrangement Conjecture

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In February 2026, Huy Tuan Pham and Lisa Sauermann (Pham and Sauermann 2026) completed, together with earlier work, a proof of Graham’s rearrangement conjecture for all sufficiently large primes. In October, Peiru Kuang and Yan Wang (Kuang and Wang 2026) extended the conclusion to sufficiently large subsets of arbitrary finite abelian groups.

The problem is easy to explain using arithmetic modulo a prime. Working modulo p means that we identify integers whose difference is divisible by p. For example, modulo 7, we have $6+4=3$.

Choose a set of distinct nonzero residues. Can we arrange them so that their running totals are all different?

For example, take

$$S=\{1,2,3,4\}$$

modulo 7. The ordering $1,2,3,4$ gives partial sums

$$1,\quad 3,\quad 6,\quad 3,$$

so a value repeats. However, the ordering $1,3,2,4$ gives

$$1,\quad 4,\quad 6,\quad 3,$$

which are all distinct.

An ordering $s_1,\dots,s_m$ of a set S is called *valid* if its nonempty partial sums

$$s_1,\quad s_1+s_2,\quad \dots,\quad s_1+\cdots+s_m$$

are pairwise distinct. In 1971, Ronald Graham (Graham 1971, 36) conjectured that every subset of the nonzero residues modulo a prime has a valid ordering.

The same question makes sense in any finite abelian group. Such a group is a finite set with an associative and commutative addition, a zero element, and an additive inverse for every element. Examples include the integers modulo any positive integer, and binary vectors with coordinatewise addition modulo 2.

The two new results can be summarized as follows.

Theorem 1 *(a) There is a constant p_0 such that, for every prime $p \geq p_0$, every subset*

$$S \subseteq (\mathbb{Z}/p\mathbb{Z}) \setminus \{0\}$$

has a valid ordering.

(b) There is an absolute positive integer m_0 such that, for every finite abelian group G, every subset

$$S \subseteq G \setminus \{0\}, \qquad |S| \geq m_0,$$

has a valid ordering.

Part (a) covers subsets of every size once the prime is sufficiently large. Part (b) covers every finite abelian group, with one universal lower bound on the size of the subset. The same m_0 works regardless of the size or structure of the group.

There is an important convention in this statement: the empty partial sum 0 is not included among the sums required to be distinct. In particular, the total sum of S is allowed to be zero. For example, the ordering $1, 2, 4$ modulo 7 has partial sums

$$1, \quad 3, \quad 0,$$

and is valid.

Progress towards Graham's conjecture initially came from opposite ends of the size range. In a 2024 preprint, Bedert and Kravitz (Bedert and Kravitz 2026) treated small sets, proving the conjecture when

$$|S| \leq \exp\left((\log p)^{1/4}\right).$$

In 2025, Bedert, Bucić, Kravitz, Montgomery and Müyesser (Bedert et al. 2025) treated large sets, including all those with

$$|S| \geq p^{1-c}$$

for an absolute constant $c > 0$.

Pham and Sauermann filled the intermediate range. More precisely, for every fixed $0 < \alpha < 1$, they proved the result whenever

$$C_\alpha \leq |S| \leq p^{1-\alpha},$$

where C_α depends only on α. Choosing $\alpha = c$ and combining the three ranges gives part (a) for all sufficiently large primes.

Why is finding a valid ordering difficult? If two partial sums agree, say

$$s_1 + \cdots + s_i = s_1 + \cdots + s_j \qquad (1 \leq i < j \leq m),$$

then cancellation gives

$$s_{i+1} + \cdots + s_j = 0.$$

Thus a collision occurs exactly when a consecutive block starting after the first term sums to zero.

A greedy approach has room to work early on. After k steps, each previously reached position forbids at most one choice for the next step. When many elements remain unused, one can avoid these forbidden choices. Near the end, however, the remaining elements may leave no such freedom. An early choice can create a difficulty that only becomes visible much later.

Pham and Sauermann use randomness to overcome this obstacle. Their proof begins with a random ordering and makes local adjustments to eliminate zero-sum segments, swapping an endpoint with a suitable later element. The essential estimates control how likely a random subset of a given size is to have any particular sum. These are called *anticoncentration estimates*: they prevent too much probability from accumulating at one value and provide the flexibility needed for the adjustments.

Kuang and Wang's extension combines probabilistic ideas with the additive structure of general abelian groups. An important ingredient is a *dissociated* subset, meaning that all its subset sums are different, including the sum of the empty subset. If S itself has this property, every ordering is valid, because different prefixes correspond to different subsets.

Their argument uses a sufficiently large dissociated subset to help organize the remaining elements while avoiding collisions. Sets without such a large subset require a different analysis, using their concentration in translates of subgroups. Together, these approaches yield the universal threshold in part (b).

The theorem does not require the elements of S to have few additive relations. A set may contain many subsets summing to zero, yet its elements can still be threaded into an ordering whose nonempty running totals are all different.

References

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