Keeping Partial Sums Small

Bogdan Grechuk · 22 Nov 2026

https://functor.network/user/3333/entry/2089

In a preprint dated September 24, 2026, OpenAI (OpenAI 2026) proved the Euclidean Steinitz–Bergström conjecture, a problem about arranging short vectors so that their partial sums stay close to the origin.

Think of vectors as instructions for a walk. Each vector tells us the direction and length of the next step. If the vectors sum to zero, the walk eventually returns to its starting point. However, it may travel very far before returning.

Even on the real line, this can happen with steps of length one. Taking a thousand steps to the right followed by a thousand steps to the left produces a large excursion. Alternating right and left keeps the walk within distance one of the origin.

What happens in higher dimensions?

For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, write

$$\|x\|_2 = \sqrt{x_1^2 + \cdots + x_d^2}$$

for its Euclidean length. Given vectors $v_1, \dots, v_N$, their partial sums are

$$v_1, \quad v_1 + v_2, \quad \dots, \quad v_1 + \cdots + v_N.$$

The new theorem gives two ways to keep these sums small.

Theorem 1 *There is an absolute constant $C > 0$ such that, for every pair of positive integers d, N and every sequence $v_1, \dots, v_N \in \mathbb{R}^d$ with $\|v_i\|_2 \le 1$, the following statements hold.*

(i) There are signs $\epsilon_1, \dots, \epsilon_N \in \{-1, 1\}$ such that

$$\left\| \sum_{i=1}^{k} \epsilon_i v_i \right\|_2 \le C\sqrt{d} \qquad (1 \le k \le N).$$

(ii) If $\sum_{i=1}^{N} v_i = 0$, there is a permutation π of $\{1, \dots, N\}$ such that

$$\left\| \sum_{i=1}^{k} v_{\pi(i)} \right\| \le C\sqrt{d} \qquad (1 \le k \le N).$$

In the first statement, we keep the prescribed order but may reverse individual steps. In the second, we keep every step unchanged but may choose their order.

The same constant works in every dimension and for every number of vectors. In particular, the bound is independent of N. Also, part (i) gives one choice of signs that works for every partial sum simultaneously. These signs may depend on the entire sequence.

The rearrangement problem has a long history. In 1913, Steinitz (Steinitz 1913) proved that a bound depending only on the dimension exists. In 1980, Grinberg and Sevast'yanov (Grinberg and Sevast'yanov 1980) established the bound d, valid for every norm on $\mathbb{R}^d$. The Euclidean Steinitz–Bergström conjecture predicted the much smaller bound $C\sqrt{d}$ for the Euclidean norm.

The square-root dependence cannot be improved. For the signed statement, take the standard coordinate vectors $e_1, \dots, e_d$. Whatever signs we choose,

$$\|\epsilon_1 e_1 + \cdots + \epsilon_d e_d\|_2 = \sqrt{d}.$$

For the rearrangement statement, consider the $d+1$ unit vectors $u_1, \dots, u_{d+1}$ pointing towards the vertices of a regular simplex centred at the origin. They sum to zero, and

$$\langle u_i, u_j \rangle = -\frac{1}{d} \qquad (i \neq j),$$

where $\langle x, y \rangle = \sum_{r=1}^{d} x_r y_r$ is the usual inner product.

Consequently, the sum of any k distinct vectors has squared length

$$k - \frac{k(k-1)}{d} = \frac{k(d+1-k)}{d}.$$

Taking $k = \lfloor (d+1)/2 \rfloor$, we obtain a length of at least $\sqrt{d}/2$. Every ordering therefore has a partial sum this large. Together with the theorem, this determines the growth of the best rearrangement bound up to absolute constant factors.

There is a particularly nice connection between the two parts of the theorem. A transference principle of Chobanyan (Chobanyan 1994) relates signing and rearrangement. Here is the short argument used in the new paper.

Start with a zero-sum family. Among all its orderings, choose one that minimizes the largest norm of a partial sum, and call this minimum M. Such an ordering exists because there are only finitely many permutations. Relabel the vectors in this order, and put

$$A_j = \sum_{i=1}^{j} v_i.$$

Thus $\|A_j\|_2 \le M$ for every j.

Apply part (i) to this ordering. Writing

$$B_j = \sum_{i=1}^{j} \epsilon_i v_i,$$

we have $\|B_j\|_2 \le R$, where $R = C\sqrt{d}$.

Let P_j be the sum of the vectors among $v_1, \dots, v_j$ carrying a positive sign, and Q_j the sum of those carrying a negative sign. Then

$$P_j = \frac{A_j + B_j}{2}, \qquad Q_j = \frac{A_j - B_j}{2}.$$

The triangle inequality gives

$$\|P_j\|_2,\ \|Q_j\|_2 \le \frac{M + R}{2}.$$

Now form a new ordering: first list the positive-sign vectors in their original order, then the negative-sign vectors in reverse order. During the first part, each partial sum is some P_j. During the second part, the unused vectors sum to some Q_j; because the total sum is zero, the current partial sum is $-Q_j$.

Every partial sum in the new ordering therefore has norm at most $(M + R)/2$. But M was the smallest possible maximum, so

$$M \le \frac{M + R}{2},$$

which implies $M \le R = C\sqrt{d}$. This proves part (ii) from part (i), with the same constant.

The difficult step is therefore the signed statement: finding one choice of signs that controls all the partial sums, however long the sequence is. The proof combines probability and convex geometry to obtain this simultaneous control.

Arbitrarily long lists of vectors can thus be controlled at the same scale forced by simple geometric examples. The number of steps disappears from the bound; only the square root of the dimension remains.

References

Chobanyan, Sergej. 1994. “Convergence a.s. Of Rearranged Random Series in Banach Space and Associated Inequalities.” In *Probability in Banach Spaces, 9*, vol. 35. Progress in Probability. Birkhäuser Boston. https://doi.org/10.1007/978-1-4612-0253-0_1.

Grinberg, V. S., and S. V. Sevast’yanov. 1980. “Value of the Steinitz Constant.” *Functional Analysis and Its Applications* 14 (2): 125–26. https://doi.org/10.1007/BF01086559.

OpenAI. 2026. *The Euclidean Steinitz–Bergström Theorem*. https://github.com/openai/math/blob/main/preprints/The-Euclidean-Steinitz-Bergstrom-theorem-

September-24-2026/The-Euclidean-Steinitz-Bergstrom-theorem-September-24-2026.pdf.
Steinitz, Ernst. 1913. “Bedingt Konvergente Reihen Und Konvexe Systeme.” *Journal für Die Reine Und Angewandte Mathematik* 143: 128–75. https://doi.org/10.1515/crll.1913.143.128.