The Generalized Riemann Hypothesis Is Proved

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On November 20, 2026, Levent Alpöge (Alpöge 2026) announced that, working with Claude, he had found a proof of the Riemann Hypothesis and its generalization to Dirichlet L-functions. The argument was released as a formal proof in the Lean proof assistant, accompanied by a short note describing its origins.

In my earlier post, we discussed the quasi-Riemann hypothesis, which keeps the nontrivial zeros a fixed positive distance away from the line $\mathrm{Re}(s) = 1$. The new announcement concerns the much stronger conclusion that all these zeros lie exactly on the line $\mathrm{Re}(s) = 1/2$.

Let us recall the functions involved. For a complex number s with $\mathrm{Re}(s) > 1$, the *Riemann zeta function* is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Analytic continuation extends it to the complex plane, apart from a pole at $s = 1$. Its zeros at the negative even integers are called trivial. The remaining zeros lie in the *critical strip*

$$0 < \mathrm{Re}(s) < 1.$$

Riemann conjectured in 1859 that every such zero has real part $1/2$.

To describe the generalization, fix a positive integer q. A *Dirichlet character modulo* q is a function $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying

$$\chi(n+q) = \chi(n), \qquad \chi(mn) = \chi(m)\chi(n), \qquad \chi(1) = 1,$$

and $\chi(n) = 0$ exactly when $\gcd(n, q) > 1$.

For example, there is a character modulo 4 which is zero on even integers, equals 1 on integers congruent to 1 modulo 4, and equals -1 on integers congruent to 3 modulo 4.

The corresponding *Dirichlet L-function* is initially defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \qquad \operatorname{Re}(s) > 1.$$

For the character just described, this becomes

$$L(s,\chi) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots .$$

These functions also extend by analytic continuation to the complex plane, with a possible pole at $s = 1$.

The announced theorem is the Generalized Riemann Hypothesis for this family.

Theorem 1 *For every positive integer q and every Dirichlet character χ modulo q, every zero s of $L(s,\chi)$ in the critical strip satisfies*

$$\operatorname{Re}(s) = \frac{1}{2}.$$

When $q = 1$, the only character is $\chi(n) = 1$, and $L(s,\chi) = \zeta(s)$. Thus the theorem includes the classical Riemann Hypothesis.

Why should the zeros of these complex functions matter for prime numbers? The connection begins with the Euler product

$$L(s,\chi) = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \qquad \operatorname{Re}(s) > 1.$$

Characters distinguish residue classes, so their L-functions give access to primes in individual arithmetic progressions.

Write $\pi(x; q, a)$ for the number of primes $p \le x$ satisfying $p \equiv a \pmod{q}$. Assume that $\gcd(a, q) = 1$. The prime number theorem for arithmetic progressions predicts an approximately equal share of the primes in each of the $\phi(q)$ eligible residue classes, where $\phi(q)$ counts the integers between 1 and q that are coprime to q.

The generalized hypothesis gives a precise error bound (Davenport 2000, chaps. 19–20):

$$\pi(x; q, a) = \frac{\operatorname{Li}(x)}{\phi(q)} + O\!\left(\sqrt{x}\log x\right), \qquad 1 \le q \le x,$$

where

$$\operatorname{Li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t}.$$

Here the O notation means that the absolute error is at most $C\sqrt{x}\log x$ for an absolute constant C, independent of q and a.

For example, the primes congruent to 1 modulo 4 and those congruent to 3 modulo 4 each have main term $\mathrm{Li}(x)/2$, with the stated error. More generally, for every fixed $0 < \epsilon < 1/2$, the estimate gives asymptotically equal shares uniformly for all moduli $q \le x^{1/2-\epsilon}$. The modulus can therefore grow substantially with the range in which we count primes.

There are also concrete consequences for gaps between ordinary primes. Carneiro, Milinovich and Soundararajan (Carneiro et al. 2019, Theorem 5) proved under the Riemann Hypothesis that, for every real $x \ge 4$, the interval

$$\left[x,\ x + \frac{22}{25}\sqrt{x}\log x\right]$$

contains a prime. Combining their theorem with the announced result removes its unproved hypothesis.

What is the idea behind the new proof? Alpöge's note traces it to Stepanov's method for the analogue of the Riemann Hypothesis over finite fields. In that setting, one constructs auxiliary functions with zeros of high multiplicity and compares the resulting lower and upper bounds on their numbers of zeros. Alpöge describes replacing this use of multiplicity by long chains of *theta lifts*, transformations between certain spaces of functions associated with different groups. The note gives an indication of the strategy; a conventional exposition of the complete argument remains to be written.

The formal proof has already undergone substantial independent checking. As of November 21, Andrew Sutherland (Sutherland 2026) reports that the released source compiles, that the headline theorems depend only on Lean's standard logical axioms, and that several independently implemented proof checkers have accepted the argument. His report also records successful comparison of the theorem statements with separately prepared statements using the standard definitions. These checks address formal correctness, while explaining and simplifying the mathematics remains a substantial task.

The attraction of the theorem is the connection it makes precise: the location of zeros on one vertical line controls the accuracy with which we can count primes. Its Dirichlet generalization gives this control throughout the arithmetic progressions, extending Riemann's original question from the primes as a whole to their distribution among residue classes.

References

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