Unforced Navier–Stokes Blowup

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On November 20, 2026, Levent Alpöge and Tristan Buckmaster (Alpöge and Buckmaster 2026) announced a construction of finite-time singularities for the three-dimensional Navier–Stokes equations without an external force. In my earlier post, we discussed a construction in which a smooth external force starts a fluid moving from rest. The new result addresses the question left open there: can a smooth initial motion produce a singularity on its own?

The Navier–Stokes equations describe the motion of an incompressible viscous fluid. Write $u(x,t) \in \mathbb{R}^3$ for its velocity at position $x \in \mathbb{R}^3$ and time $t \geq 0$, and $p(x,t) \in \mathbb{R}$ for its pressure. With no external force, the equations are

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = \nu \Delta u - \nabla p, \qquad \nabla \cdot u = 0.$$

Here $\nu > 0$ is the viscosity, ∇p is the pressure gradient, and Δ is the Laplacian, applied to each component of u. The condition $\nabla \cdot u = 0$ expresses incompressibility. The nonlinear term $(u \cdot \nabla)u$ describes transport by the fluid's own motion, while $\nu \Delta u$ describes the smoothing effect of viscosity.

We prescribe an initial velocity by $u(x,0) = u^0(x)$. It must be *divergence-free*, meaning $\nabla \cdot u^0 = 0$. We also require uniformly bounded kinetic energy:

$$\sup_{t \geq 0} \int_{\mathbb{R}^3} |u(x,t)|^2 \, \mathrm{d}x < \infty.$$

The integral is twice the kinetic energy when the density of the fluid is one.

The announced result can be stated as follows.

Theorem 1 *For every viscosity $\nu > 0$, the unforced three-dimensional Navier–Stokes equations have the following properties.*

1. *There exists a smooth, divergence-free initial velocity u^0 on $\mathbb{R}^3$, zero outside a bounded region, for which there is no solution smooth for all $t \geq 0$ with uniformly bounded kinetic energy.*
2. *There exists a smooth, divergence-free initial velocity u^0 that is 1-periodic in each coordinate, for which there is no solution smooth for all $t \geq 0$ whose velocity remains 1-periodic in each coordinate.*

The periodic case can be viewed as a fluid in a unit cube with opposite faces identified: everything repeats from one cube to the next. Its kinetic energy is measured over one cube.

For smooth initial data of these kinds, smooth solutions exist for a short time. The theorem says that some initial data prevent this smooth evolution from continuing forever. A *finite-time singularity* occurs at a time $T < \infty$ if the solution is smooth for $0 \le t < T$ but cannot be extended smoothly through T.

These are precisely the two breakdown alternatives in Fefferman's formulation of the Navier–Stokes Millennium Prize Problem (Fefferman 2000), with the external force set equal to zero.

The construction also describes how a singularity develops. One idea, going back to Leray (Leray 1934), is to look for a flow that keeps its shape when observed at a shrinking spatial scale. Near a prospective singularity at the origin and time T, introduce

$$y = \frac{x}{\sqrt{T - t}}, \qquad s = -\log(T - t).$$

As t approaches T, the new time s tends to infinity. We observe the rescaled velocity $\sqrt{T - t}\, u(x, t)$ in these variables.

A *backward self-similar* flow would have a rescaled profile $U(y)$ independent of s. This attractive possibility encountered a serious obstruction. Nečas, Růžička and Šverák (Nečas et al. 1996), followed by Tsai (Tsai 1998), proved that such a profile must be zero if

$$\int_{\mathbb{R}^3} |U(y)|^q \, \mathrm{d}y < \infty$$

for some finite $q \ge 3$.

Alpöge and Buckmaster instead use a profile that changes with rescaled time but repeats after a fixed interval:

$$W(y, s + S) = W(y, s) \qquad \text{for some } S > 0.$$

This is called *backward discrete self-similarity*. In the original variables, the same pattern returns at a geometric sequence of smaller spatial scales as the singular time approaches.

There is a further difficulty: a model profile need not itself give an initial velocity that vanishes outside a bounded region, or one that is spatially periodic. The authors construct solutions with these required initial conditions and show that, in rescaled variables, they approach the repeating profile.

The resulting flows have a quantitative blowup rate. On $\mathbb{R}^3$, there are positive constants c and C such that

$$c(T-t)^{-1/4} \le \left(\int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \left| \frac{\partial u_i}{\partial x_j}(x,t) \right|^2 \, \mathrm{d}x \right)^{1/2} \le C(T-t)^{-1/4}.$$

Thus the spatial variation of the velocity becomes unbounded while the total kinetic energy remains bounded. The periodic construction has the same rate, with the integral taken over one period cell.

The proof combines analysis and computation. The authors first find a numerical approximation to the repeating profile. A certificate consists of finite data satisfying explicit inequalities that allow this approximation to be corrected to an exact solution. The authors report checking these inequalities using interval arithmetic, which keeps rigorous bounds on rounding errors. The accompanying Lean formalization proves the deduction of breakdown from certificate conditions.

There is an important qualification concerning the current release. As of November 21, 2026, the repository does not include the numerical certificate or the interval-arithmetic computation used to check it. It also records that the patched Lean package had not been rebuilt when the repository was assembled. The result should therefore be described as an announced breakthrough awaiting independent verification.

Finally, singularity formation is compatible with Leray’s classical existence theorem (Leray 1934). Leray constructed global weak solutions, in which the equations are interpreted after integration against smooth test functions. Such solutions can continue beyond the time when smoothness fails. The new announcement concerns the much more demanding question of whether a fluid starting with a completely smooth velocity must remain smooth forever.

References

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