Kissing Numbers: From Pairs to Triples

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https://functor.network/user/3333/entry/2086

This post continues my series on favourite theorems of the twenty-first century. For an overview of the categories and my earlier selections, see this introductory post.

My choice for 2006 in Geometry and Topology is a theorem of Christine Bachoc and Frank Vallentin giving new upper bounds for kissing numbers. Their paper was first posted online on August 16, 2006, and appeared in the *Journal of the American Mathematical Society* in 2008 (Bachoc and Vallentin 2008).

How many balls of the same size can touch one central ball? We require all the balls to have disjoint interiors, but they may touch one another. The largest possible number of balls surrounding the central one in $\mathbb{R}^d$ is called the *kissing number*, denoted by $k(d)$. In the plane, six equal coins fit around a seventh, and there is no room for an eighth. Thus $k(2) = 6$. In three dimensions, the answer is 12. In four dimensions, Oleg Musin (Musin 2008) proved that $k(4) = 24$. His proof first appeared as a preprint in 2003, so the four-dimensional problem had already been solved by the time Bachoc and Vallentin posted their paper. Dimension five was therefore the first dimension in which the kissing number was still unknown. The exact value $k(8) = 240$ was also already known (Odlyzko and Sloane 1979), which explains why dimension eight is absent from the table below.

To see how this becomes a problem about points on a sphere, take the central ball to have radius one and centre zero. The centres of the surrounding balls lie at distance two from zero. Dividing these centres by two gives unit vectors, any two of which must be at distance at least one. For unit vectors x and y,

$$\|x - y\|^2 = 2 - 2\langle x, y\rangle,$$

so the condition is equivalent to $\langle x, y\rangle \leq 1/2$. In other words, $k(d)$ is the largest number of unit vectors in $\mathbb{R}^d$ with angles of at least 60 degrees between every pair.

For many years, the main tool for bounding this number was Delsarte's linear programming method. It places restrictions on how often different inner products can occur among pairs of vectors. Using this method, Odlyzko and Sloane (Odlyzko and Sloane 1979) obtained, among other results, the bounds $k(5) \leq 46$ and $k(6) \leq 82$.

Bachoc and Vallentin's idea was to look at triples of vectors. This gave the following improvements.

Theorem. In each dimension listed below, the kissing number is at most the number in the last row.

d	5	6	7	9	10
Previous upper bound	46	82	140	379	594
Bachoc–Vallentin bound	45	78	135	366	567

What can triples tell us that pairs cannot? Take three unit vectors and write a, b, c for their three pairwise inner products. The matrix of their inner products is

$$\begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix}.$$

It is positive semidefinite: its quadratic form is the squared length of a linear combination of the three vectors. In particular, its determinant is nonnegative, which gives

$$1 + 2abc - a^2 - b^2 - c^2 \geq 0.$$

This is a restriction on how the three inner products can occur together.

The proof goes beyond checking individual triples. It uses polynomials adapted to the sphere to derive positive semidefinite matrix constraints involving sums over *all* triples in a configuration. These constraints lead to a semidefinite program. Since every arrangement of touching balls must satisfy the constraints, the program gives an upper bound valid for all arrangements. The authors used numerical computations to find certificates for the bounds and then checked them using rational arithmetic.

In my earlier post on upper bounds for sphere packings, we saw how linear programming could bound the density of a packing. What I like about Bachoc and Vallentin's theorem is that it shows how to bring more geometry into an optimization problem: keeping track of which angles occur together lets us rule out configurations that the pairwise bounds could not exclude. The method could also be refined. In 2010, Mittelmann and Vallentin (Mittelmann and Vallentin 2010) lowered the upper bounds in dimensions five and seven to 44 and 134, respectively. Machado and de Oliveira Filho (Machado and Oliveira Filho 2018) used symmetry to make larger semidefinite programs tractable, obtaining

$k(9) \le 363$ and $k(10) \le 553$. In 2024, de Laat, Leijenhorst, and de Muinck Keizer (Laat et al. 2024) used a method that also takes four-point configurations into account to prove $k(6) \le 77$. Progress came from the other direction as well: Ganzhinov (Ganzhinov 2025) constructed an arrangement of 510 touching balls in dimension ten, improving the previous lower bound of 500.

There has been another round of improvements in November 2026. Tomar (Tomar 2026) gave exact rational certificates for $k(5) \le 43$, $k(6) \le 76$, and $k(7) \le 132$. A separate preprint by Nyam-Osor (Nyam-Osor 2026) reports $k(10) \le 552$, one below the journal-published bound of 553.

Here are the best bounds I found as of November 20, 2026, including those reported in the new preprints. The lower bounds come from explicit constructions; see also Henry Cohn's table of kissing number bounds.

d	5	6	7	9	10
Lower bound for $k(d)$	40	72	126	306	510
Upper bound for $k(d)$	43	76	132	363	552

None of these five kissing numbers is known exactly.

Bachoc and Vallentin's theorem is still my choice for 2006 because it gave later researchers a way to keep improving the bounds. Once the geometry of triples was expressed in a semidefinite program, one could use symmetry to solve larger instances or bring four points into the picture. The idea also reached beyond kissing numbers: Cohn, de Laat, and Salmon (Cohn et al. 2022) used three-point constraints to improve upper bounds on sphere-packing density. That reach is what makes the theorem memorable to me: an idea introduced to count the balls touching one ball also helps bound how densely balls can fill an entire space.

References

Bachoc, Christine, and Frank Vallentin. 2008. "New Upper Bounds for Kissing Numbers from Semidefinite Programming." *Journal of the American Mathematical Society* 21 (3): 909–24. https://doi.org/10.1090/S0894-0347-07-00589-9.

Cohn, Henry, David de Laat, and Andrew Salmon. 2022. *Three-Point Bounds for Sphere Packing*. https://arxiv.org/abs/2206.15373.

Ganzhinov, Mikhail. 2025. "Highly Symmetric Lines." *Linear Algebra and Its Applications* 722: 12–37. https://doi.org/10.1016/j.laa.2025.05.002.

Laat, David de, Nando M. Leijenhorst, and Willem H. H. de Muinck Keizer. 2024. *Optimality and Uniqueness of the D_4 Root System*. https://arxiv.org/abs/2404.18794.

Machado, Fabrício Caluza, and Fernando Mário de Oliveira Filho. 2018. “Improving the Semidefinite Programming Bound for the Kissing Number by Exploiting Polynomial Symmetry.” *Experimental Mathematics* 27 (3): 362–69. https://doi.org/10.1080/10586458.2017.1286273.

Mittelmann, Hans D., and Frank Vallentin. 2010. “High-Accuracy Semidefinite Programming Bounds for Kissing Numbers.” *Experimental Mathematics* 19 (2): 175–79. https://doi.org/10.1080/10586458.2010.10129070.

Musin, Oleg R. 2008. “The Kissing Number in Four Dimensions.” *Annals of Mathematics* 168 (1): 1–32. https://doi.org/10.4007/annals.2008.168.1.

Nyam-Osor, Uchral. 2026. *The Kissing Number in Dimension Ten Is at Most 552*. https://doi.org/10.5281/zenodo.24187437.

Odlyzko, Andrew M., and Neil J. A. Sloane. 1979. “New Bounds on the Number of Unit Spheres That Can Touch a Unit Sphere in n Dimensions.” *Journal of Combinatorial Theory, Series A* 26 (2): 210–14. https://doi.org/10.1016/0097-3165(79)90074-8.

Tomar, Tejasvi Singh. 2026. *The Kissing Numbers in Dimensions Five, Six and Seven Are at Most 43, 76 and 132*. https://doi.org/10.5281/zenodo.24093811.