Crouzeix’s Conjecture: Why the Constant Is Two

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On August 4, 2026, Emiel Lorist and Felix L. Schwenninger (Lorist and Schwenninger 2026) posted a proof of Crouzeix’s conjecture. Shanmu Jin (Jin 2026) had posted an independent proof shortly before. The conjecture asks how much a polynomial can amplify a matrix, and its sharp constant, 2, can already be seen in a simple 2×2 example.

Let A be a complex square matrix. If $p(z) = a_0 + a_1 z + \cdots + a_k z^k$, we define

$$p(A) = a_0 I + a_1 A + \cdots + a_k A^k.$$

We measure the size of $p(A)$ by its operator norm, the largest length of $p(A)u$ over all unit vectors u.

For a normal matrix, such as a Hermitian matrix, the spectral theorem gives a simple answer: $\|p(A)\|$ is the largest value of $|p(\lambda)|$ over the eigenvalues λ of A. For a general matrix, the eigenvalues alone may tell us very little. For example, the matrix

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

has both eigenvalues equal to zero, yet $\|A\| = 2$.

Crouzeix’s idea was to use a larger set, the *numerical range*:

$$W(A) = \{u^* A u : u^* u = 1\},$$

where u^* denotes the conjugate transpose of u. This is a compact convex subset of the complex plane containing every eigenvalue of A. For the matrix above, a unit vector $u = (a, b)$ gives $u^* A u = 2\overline{a}b$. Its modulus is at most $|a|^2 + |b|^2 = 1$, and equality is attained when $a = b = 1/\sqrt{2}$. Thus, for $p(z) = z$, the norm of $p(A)$ is exactly twice the maximum of $|p|$ on $W(A)$.

Could the ratio ever be larger than 2? In 2004, Michel Crouzeix (Crouzeix 2004) conjectured that it could not. He proved a universal bound of 11.08 in 2007 (Crouzeix 2007). In 2017, Crouzeix and Palencia (Crouzeix and Palencia 2017) lowered it to $1 + \sqrt{2} \approx 2.414$. The 2026 proofs closed the remaining gap.

Theorem. For every complex square matrix A and every polynomial p with complex coefficients,

$$\|p(A)\| \leq 2 \max_{z \in W(A)} |p(z)|.$$

The constant 2 is best possible.

Lorist and Schwenninger proved the inequality more generally for bounded operators on complex Hilbert spaces, with the maximum replaced by a supremum. Their theorem also allows rational functions whose poles lie outside the closure of the numerical range.

The key idea in their proof is to use all the powers of a function at once. An earlier integral formula compares $p(A)$ with a weighted average of the values of p on the boundary of a convex region surrounding $W(A)$. There is an error term, and estimating it directly loses something in the constant. Applying the formula to $p, p^2, p^3, \ldots$ instead gives a sequence of bounded errors with a useful commutation property. A telescoping argument then forces the sharp bound.

There is a natural further question. Suppose $P(z)$ is a matrix whose entries are polynomials. Evaluating each entry at A gives a block matrix $P(A)$. Does the same bound

$$\|P(A)\| \leq 2 \max_{z \in W(A)} \|P(z)\|$$

hold regardless of how many rows and columns P has? This is the *complete Crouzeix conjecture*. OpenAI's October release gave a proof (OpenAI 2026), and the complete inequality has also been formalized in Lean.

References

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