Avoiding Infinite Geometric Progressions

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https://functor.network/user/3333/entry/2084

On October 6, 2026, OpenAI released a preprint (OpenAI 2026) settling the geometric progression case of the Erdős similarity conjecture. On November 2, Alex Iosevich and Alexia Yavicoli (Iosevich and Yavicoli 2026) posted an exposition of the proof using finite probability and basic measure theory. Let me explain the problem and what makes this case interesting.

Suppose that $E \subseteq \mathbb{R}$ is a measurable set of positive length. Given a pattern $A \subseteq \mathbb{R}$, must E contain a translated and rescaled copy of A? In other words, must there be real numbers x and $s \neq 0$ such that

$$x + sA = \{x + sa : a \in A\} \subseteq E?$$

A negative value of s also allows us to reflect the pattern.

For every finite pattern, the answer is yes. To see this, first replace E by a bounded subset of positive measure if necessary. As s tends to zero, each translate $E - sa$ overlaps with nearly all of E, because translations are continuous in measure. Since there are only finitely many $a \in A$, for sufficiently small nonzero s all these translates have a point x in common. Then $x + sa \in E$ for every $a \in A$.

In 1974, Paul Erdős (Erdős 1974) asked whether this property always fails for infinite patterns. More precisely, the *Erdős similarity conjecture* says that for every infinite set $A \subseteq \mathbb{R}$, there is a measurable set of positive length containing no affine copy of A.

An early result of Kenneth Falconer (Falconer 1984) showed that one can avoid slowly decreasing sequences such as $\{1/n : n \geq 1\}$. The consecutive terms of $1/n$ have ratios tending to one. What if the ratio instead stays fixed, as in the sequence $1/2, 1/4, 1/8, \ldots$? Even this simple case resisted proof for decades.

Theorem. For every $0 < q < 1$ and every $0 < \eta < 1$, there is a compact set $E \subseteq [0,1]$ of Lebesgue measure greater than $1 - \eta$ such that, for every $x \in \mathbb{R}$ and every $s \neq 0$, at least one of the points

$$x + sq, \quad x + sq^2, \quad x + sq^3, \quad \ldots$$

lies outside E.

For example, one can keep more than 99% of the unit interval and still avoid every affine copy of the halving sequence. By the finite-pattern argument, the set contains copies of the first N terms for every N. Yet no single choice of translation and scale works for the whole sequence. The preprint treats every fixed ratio q, and the release also includes a Lean formalization for $q = 1/2$.

The proof starts by asking what to remove. The aim is to find a small open set that meets every copy of the progression. If we choose tiny intervals at random, a given copy has many chances to hit one of them. The difficulty is that we must deal with all translations and all scales. A single fine grid creates too many possible configurations to control with this simple argument.

The key idea is to use nested grids and arrange the random choices in a finite tree. Each group of tests depends on only a small part of the tree, so there are fewer scale configurations to check. The argument first handles scales between 1 and 2; rescaling and reflecting then handle all nonzero scales. There is also a useful final trick: if x is an exceptional base point, removing a small open neighbourhood of x catches the tail of every copy based there, since $x + sq^n$ tends to x.

The theorem settles the conjecture for geometric progressions. The question for arbitrary infinite sets remains open.

References

Erdős, Paul. 1974. “Problems.” *Mathematica Balkanica* 4: 203–4.

Falconer, Kenneth J. 1984. “On a Problem of Erdős on Sequences and Measurable Sets.” *Proceedings of the American Mathematical Society* 90 (1): 77–78. https://doi.org/10.1090/S0002-9939-1984-0722418-0.

Iosevich, Alex, and Alexia Yavicoli. 2026. *The Geometric Case of the Erdős Similarity Conjecture. A Human Exposition of the OpenAI Proof*. https://arxiv.org/abs/2611.02454.

OpenAI. 2026. *The Geometric Case of the Erdős Similarity Conjecture*. https://github.com/openai/math/blob/main/preprints/The-geometric-case-of-the-Erdos-similarity-conjecture-October-5-2026/geometric-erdos-similarity.pdf.