Falconer’s Distance Conjecture in All Dimensions

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In July, I wrote about Falconer’s distance set problem in the plane, focusing on a breakthrough by Larry Guth, Alex Iosevich, Yumeng Ou, and Hong Wang. OpenAI’s October 2026 mathematics release includes a preprint (OpenAI 2026) proving the full conjecture in every dimension. The theorem has also been formalized in Lean.

Let us recall the problem. For a compact set $E \subseteq \mathbb{R}^d$, its *distance set* is

$$\Delta(E) = \{|x-y| : x, y \in E\},$$

where $|x-y|$ is the Euclidean distance between x and y. In the finite Erdős distinct distances problem, we ask how many different distances a set of points must determine. Falconer’s question is about a different kind of size: when must the set of distances have positive Lebesgue measure, that is, positive length?

The size of E itself is measured by its *Hausdorff dimension*, denoted by $\dim_H E$. To define it, cover E by countably many balls of radii $r_i \le \delta$ and take the infimum of $\sum_i r_i^s$ over all such covers. The Hausdorff dimension is the infimum of the exponents s for which this quantity tends to zero as $\delta \to 0$. It can be fractional: a compact set in the plane can have Hausdorff dimension strictly between 1 and 2 while having zero area.

In 1985, Kenneth Falconer (Falconer 1985) proved that $\dim_H E > (d+1)/2$ guarantees that $\Delta(E)$ has positive length. He also constructed a compact set of dimension $d/2$ whose distance set has length zero. The idea behind his example comes from a grid. A d-dimensional grid with q points along each axis and spacing $1/q$ contains q^d points, but every squared distance is an integer multiple of $1/q^2$ between 0 and d. Thus there are at most dq^2+1 distinct distances. If we replace each grid point by a ball of radius much smaller than q^{-2}, these distances spread into intervals of small total length. Falconer’s construction turns this observation into a fractal example by using grids at successively finer scales.

Could a set of dimension even slightly greater than $d/2$ still determine a set of distances of length zero? Falconer's conjecture predicted that it could not. In the plane, Guth, Iosevich, Ou, and Wang lowered the sufficient threshold to $5/4$ (Guth et al. 2020). In dimensions $d \geq 3$, Du, Ou, Ren, and Zhang brought it down to $d/2 + 1/4 - 1/(8d+4)$ (Du et al. 2023). The new result reaches $d/2$ itself as the threshold.

Theorem. Let $d \geq 2$ be an integer and let $E \subseteq \mathbb{R}^d$ be compact. If $\dim_H E > d/2$, then $\Delta(E)$ has positive Lebesgue measure.

Thus, in the plane, a compact set of Hausdorff dimension 1.01 already determines a set of distances of positive length. The inequality in the theorem must be strict, as Falconer's example shows. The conclusion uses all pairs of points in E. Asking whether a single point $x \in E$ determines a positive-length set $\{|x-y| : y \in E\}$ at the same threshold is a stronger question.

One way to understand the proof is to imagine sampling pairs of points and recording their distances. The resulting distribution might, in principle, be concentrated on a set of length zero. The proof keeps a positive mass of pairs while controlling how strongly their directions can cluster. It then uses Fourier estimates at successive scales to control pairs with nearly equal distances. The strict inequality $\dim_H E > d/2$ gives enough room to combine these estimates and obtain a nonzero measure on $\Delta(E)$ with a density with respect to length.

References

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