# How Noise Separates Singular Values

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On November 3, 2026, a paper by Sam Bednarski, Jonathan DeWitt, and Anthony Quas (Bednarski et al. 2026) was accepted for publication in *Inventiones mathematicae*. It appeared online a week later. The paper answers a natural question: what happens when we add a little independent noise to each factor in a long product of matrices?

In my post about Van Vu's talk on random matrix theory, I discussed the smallest singular value of a single random matrix. In this post we discuss matrix products, where the dimension stays fixed, the number of factors grows, and we are interested in the gaps between successive singular values.

Recall that the *singular values* of a real $d \times d$ matrix $B$ are the square roots of the eigenvalues of $B^{\mathsf{T}}B$, where $B^{\mathsf{T}}$ is the transpose of $B$. We list them in decreasing order, with repetitions:

$$s_1(B) \geq s_2(B) \geq \cdots \geq s_d(B) \geq 0.$$

Geometrically, an invertible matrix sends the Euclidean unit ball to an ellipsoid whose semiaxis lengths are its singular values. In two dimensions, the ratio $s_1(B)/s_2(B)$ measures how elongated the resulting ellipse is.

Take any fixed sequence of real $d \times d$ matrices $A_1, A_2, \ldots$ with $s_1(A_j) \leq 1$ for every $j$. Let $E_1, E_2, \ldots$ be random matrices whose entries are all independent standard normal variables. For $0 < \epsilon \leq 1$, form the product

$$B_n = (A_n + \epsilon E_n) \cdots (A_1 + \epsilon E_1).$$

Thus $\epsilon$ is the standard deviation of the noise added to each entry.

For uniform noise, Atnip, Froyland, González-Tokman, and Quas (Atnip et al. 2026) showed that the ratios of successive singular values grow at least exponentially, with a positive rate that can be guaranteed for every original sequence. They asked whether the guaranteed rate could be bounded below by a constant times $\epsilon^2$. The new paper obtains a quadratic bound for a broad class of noise distributions. Here is the Gaussian case.

**Theorem.** For each $d \geq 2$ there is a constant $c_d > 0$ such that, for every sequence $A_1, A_2, \ldots$ and every $\epsilon$ as above, with probability one all the products $B_n$ are invertible and

$$\liminf_{n\to\infty} \frac{1}{n} \log \frac{s_k(B_n)}{s_{k+1}(B_n)} \geq c_d \epsilon^2, \qquad k = 1, \ldots, d-1.$$

In particular, for each fixed $\epsilon$, almost surely every ratio $s_k(B_n)/s_{k+1}(B_n)$ is at least $\exp(c_d \epsilon^2 n/2)$ for all sufficiently large $n$. This is a statement about relative size: two singular values can both tend to zero while their ratio grows exponentially. The same constant $c_d$ works however the original matrices vary.

The identity matrix gives a striking example. If every $A_j$ is the identity, then without noise the product is always the identity and all its singular values are $1$. Yet even arbitrarily small Gaussian perturbations make every successive ratio grow exponentially. In this example, Bednarski and Quas (Bednarski and Quas 2025) showed that the normalized logarithmic gap converges almost surely as $n \to \infty$, and that its limit is $\epsilon^2 + o(\epsilon^2)$ as $\epsilon \to 0^+$. Thus the quadratic order is best possible for a bound that must work for every sequence.

Let me briefly describe an idea behind the proof. To study the $k$th gap, the authors start with nested subspaces of dimensions $k-1$ and $k+1$. Choosing a $k$-dimensional subspace between them amounts to choosing a line in a two-dimensional space. The matrix product changes how those lines are spaced around a circle of directions, and the authors relate this distortion to the logarithmic singular value gap. Building on the relative entropy method of Gorodetski and Kleptsyn (Gorodetski and Kleptsyn 2026), they obtain a contribution of at least a constant times $\epsilon^2$ per factor. These contributions accumulate over the product.

## References


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