The Unit Sphere Determines the Whole Space

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In September 2026, OpenAI released a preprint (OpenAI 2026) announcing a positive solution to *Tingley's problem*, a famous question in the geometry of Banach spaces that had remained open since 1987.

The result has a remarkably simple interpretation: *knowing all the distances between points on the unit sphere of a Banach space is sufficient to reconstruct the entire space, including its linear structure.*

Let us recall the definitions.

A *Banach space* is a complete normed vector space. Thus, it is a vector space X equipped with a function $\|\cdot\| : X \to [0,\infty)$ satisfying

$$\|x\| = 0 \iff x = 0, \qquad \|\lambda x\| = |\lambda|\|x\|,$$

and

$$\|x+y\| \le \|x\| + \|y\|,$$

such that every Cauchy sequence converges.

For example, $\mathbb{R}^n$ is a Banach space with the usual Euclidean norm

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

But there are many other norms, such as

$$\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}.$$

With the Euclidean norm, the unit sphere in $\mathbb{R}^2$ is the familiar circle. With the maximum norm, the unit sphere is the boundary of a square.

In general, the *unit sphere* of a Banach space X is

$$S_X = \{x \in X : \|x\| = 1\}.$$

An *isometry* between two metric spaces is a map preserving distances. Thus, a map

$$f : S_X \to S_Y$$

between the unit spheres of two Banach spaces is an isometry if

$$\|f(x) - f(y)\| = \|x - y\| \qquad \text{for all } x, y \in S_X.$$

Suppose that f is also surjective, meaning that every point of S_Y is the image of some point of S_X.

Does such a map necessarily extend to a linear isometry between the entire spaces?

To see why this question is natural, let us first consider ordinary Euclidean spaces.

Suppose that f is a surjective isometry of the Euclidean unit sphere S^{n-1} onto itself.

For unit vectors x, y, we have

$$\|x - y\|^2 = 2 - 2x \cdot y,$$

where $x \cdot y$ denotes the usual scalar product.

Since f preserves distances, it also preserves scalar products:

$$f(x) \cdot f(y) = x \cdot y.$$

It follows that f is the restriction of an orthogonal linear transformation of $\mathbb{R}^n$.

In other words, every distance-preserving bijection of a Euclidean sphere is obtained by rotating or reflecting the ambient space.

For general Banach spaces, however, this elementary argument no longer works. The norm need not arise from a scalar product, so the distances between points cannot be expressed in such a simple algebraic form.

The problem is therefore much harder.

There is a classical theorem that provides some hope.

In 1932, Mazur and Ulam (Mazur and Ulam 1932) proved that every surjective isometry between real normed vector spaces is *affine*. Thus, if $F : X \to Y$ is a surjective isometry, then

$$F(x) = T(x) + b,$$

where $T : X \to Y$ is a surjective linear isometry and $b \in Y$ is fixed.

In particular, if $F(0) = 0$, then F must be linear.

This theorem is remarkable because its assumption is purely metric: F is required to preserve distances, but no linearity is assumed.

In 1972, Mankiewicz (Mankiewicz 1972) proved a stronger extension theorem, showing in particular that a surjective isometry between the closed unit balls of two real Banach spaces extends to an affine isometry of the entire spaces.

Recall that the *closed unit ball* is

$$B_X = \{x \in X : \|x\| \leq 1\}.$$

Thus, the metric structure of the unit ball determines the linear structure of the whole space.

But what if we only know the metric structure of the unit *sphere*, rather than the entire ball?

This question was raised by David Tingley in 1987 (Tingley 1987).

The main difficulty is that a sphere is much smaller than a ball: it does not contain any points of norm less than one.

Suppose that we know a surjective isometry

$$f : S_X \to S_Y.$$

There is an obvious candidate for its extension. Every nonzero vector $x \in X$ can be written uniquely as

$$x = \|x\| \frac{x}{\|x\|}.$$

We can therefore define

$$T(x) = \begin{cases} \|x\| f\left(\dfrac{x}{\|x\|}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

This map preserves norms:

$$\|T(x)\| = \|x\|.$$

It also preserves distances between points lying on the same sphere centered at the origin. Indeed, for $r > 0$ and $x, y \in S_X$,

$$\|T(rx) - T(ry)\| = r\|f(x) - f(y)\| = r\|x - y\|.$$

However, it is not obvious that T preserves distances between points having *different norms*.

In other words, how can we deduce from the equality

$$\|f(x) - f(y)\| = \|x - y\|$$

that

$$\|rf(x) - sf(y)\| = \|rx - sy\|$$

for arbitrary $r, s > 0$?

This is the central difficulty in Tingley's problem.

For a real Banach space X, we say that X has the *Mazur–Ulam property* if, for every real Banach space Y, every surjective isometry $S_X \to S_Y$ extends to a surjective linear isometry $X \to Y$.

Tingley's problem asks whether *every real Banach space has the Mazur–Ulam property*.

Many special cases were resolved over the years.

For example, Ding (Ding 2003) established extension results for ℓ^p spaces, while Mori and Ozawa (Mori and Ozawa 2020) proved the Mazur–Ulam property for important classes of operator algebras.

In 2022, Banakh (Banakh 2022) completed the proof that every two-dimensional real Banach space has the Mazur–Ulam property.

This is much less obvious than the Euclidean case: in two dimensions, the unit sphere of a general norm may have straight edges, corners, or a very complicated shape.

Nevertheless, the full problem, especially in arbitrary infinite-dimensional spaces, remained open.

The new OpenAI preprint (OpenAI 2026) announces its complete resolution.

Theorem 1 (OpenAI) *Let X and Y be nonzero real Banach spaces, and let*

$$f : S_X \to S_Y$$

be a surjective isometry.
Define $T : X \to Y$ by

$$T(0) = 0, \qquad T(x) = \|x\| f\left(\frac{x}{\|x\|}\right) \quad \text{for } x \neq 0.$$

Then T is a surjective real-linear isometry.
In particular, every surjective isometry between the unit spheres of two real Banach spaces extends uniquely to a surjective linear isometry between the spaces.

The theorem applies to Banach spaces of arbitrary dimension, including infinite-dimensional and nonseparable spaces.

There are no assumptions of smoothness, strict convexity, reflexivity, or any other special geometric property.

Thus, every real Banach space has the Mazur–Ulam property.

An important point is that the theorem applies only to *surjective* sphere isometries. It does not say that every distance-preserving embedding of one unit sphere into another must extend to a linear embedding.

Also, when the theorem is applied to complex Banach spaces considered as real spaces, it guarantees real linearity, not necessarily complex linearity.

For example, complex conjugation

$$z \longmapsto \overline{z}$$

preserves distances on the unit circle, but is not complex-linear.

Let us briefly discuss the proof.

As explained above, the natural candidate T automatically preserves distances between points having equal norms.

The authors therefore measure the possible failure of distance preservation between points at different distances from the origin.

For unit vectors $x, y \in X$ and $0 \leq t \leq 1$, consider the quantity

$$D_t(x, y) = \|f(x) - tf(y)\| - \|x - ty\|.$$

If

$$D_t(x, y) = 0$$

for every x, y and every t, then the radial extension T preserves all distances.

By the Mazur–Ulam theorem, it must then be linear.

Thus, the entire problem reduces to proving that D_t always vanishes.

The authors argue by contradiction. Suppose that the largest possible absolute value of this expression is positive.

A compactness construction, involving an *ultrapower* of Banach spaces, allows them to assume that this largest error is actually attained.

They then study certain line segments, called *chords*, joining points of the unit spheres. The condition that the error is maximal imposes strong restrictions on how these chords can be situated.

One of the main geometric ingredients is the use of *supporting functionals*. A supporting functional at a unit vector v is a linear functional φ satisfying

$$\|\varphi\| \leq 1, \qquad \varphi(v) = 1.$$

Geometrically, such a functional describes a hyperplane supporting the unit ball at v.

The authors show that maximality of the distance error forces certain chords to share supporting functionals. This property allows them to construct a closed convex set of possible chord directions, preserved by a carefully defined transformation.

The transformation need not be a contraction in the ordinary sense. Instead, it decreases a quantity measuring how far a bounded set is from being compact.

A fixed point theorem of Darbo (Darbo 1955) then provides a direction for which two extremal chords become aligned.

Finally, the authors use convexity of the norm to show that this aligned configuration is impossible. Depending on the geometry of the chords, either a small perturbation increases the supposedly maximal error, or two estimates for the same distance contradict one another.

Therefore, the maximal error must be zero, and the radial extension preserves every distance.

The Mazur–Ulam theorem completes the proof by showing that this extension is linear.

The result illustrates a striking kind of rigidity. Even though a general Banach-space unit sphere may be highly irregular and may have no obvious algebraic structure of its own, its intrinsic distances determine the norm and the linear structure of the entire ambient space.

The new paper is available as a preprint, and independent mathematical scrutiny of the proof remains important.

If confirmed, the theorem settles a problem that has attracted attention for nearly forty years, extending the classical Mazur–Ulam theory to its most natural boundary case.

The conclusion can be expressed in one sentence: *two real Banach spaces with isometric unit spheres must themselves be linearly isometric.*

References

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