Fixed Points Without Contraction

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In September 2026, OpenAI released a preprint (OpenAI 2026) announcing a proof of a longstanding conjecture in functional analysis: every nonexpansive map of a nonempty closed bounded convex subset of a reflexive Banach space into itself has a fixed point.

The statement looks very similar to Banach's famous fixed point theorem of 1922. However, a seemingly minor difference in the assumptions makes the new result much more difficult.

Let us begin with the definitions.

A *Banach space* is a complete normed vector space. In other words, it is a vector space X equipped with a norm $\|\cdot\|$, measuring the size of its elements, such that every Cauchy sequence converges.

For example, the familiar Euclidean space $\mathbb{R}^n$ is a Banach space with the usual norm

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Another important example is the space ℓ^2 of all infinite real sequences

$$x = (x_1, x_2, \dots)$$

satisfying

$$\sum_{j=1}^{\infty} x_j^2 < \infty,$$

equipped with the norm

$$\|x\| = \left(\sum_{j=1}^{\infty} x_j^2 \right)^{1/2}.$$

A point $x \in X$ is called a *fixed point* of a map $F : X \to X$ if

$$F(x) = x.$$

Banach's contraction principle, proved in 1922, says that if X is a complete metric space and $F : X \to X$ satisfies

$$\|F(x) - F(y)\| \leq c\|x - y\|$$

for all $x, y \in X$, where $0 < c < 1$, then F has a unique fixed point.

Moreover, starting from any point x_0, the sequence

$$x_{n+1} = F(x_n)$$

converges to this fixed point.

The condition $c < 1$ is essential to the elementary proof: the distances between successive iterates decrease geometrically.

But what happens if we only assume that

$$\|F(x) - F(y)\| \leq \|x - y\|?$$

Such a map is called *nonexpansive*. It never increases the distance between two points, but it need not decrease that distance either.

For example, consider

$$F(x) = 1 - x, \qquad x \in [0, 1].$$

This is a nonexpansive map of the interval into itself. It has the fixed point $x = 1/2$, but for any other starting point the iterates alternate between x and $1 - x$, so they do not converge.

Thus, even when a fixed point exists, the iterative conclusion of Banach's theorem can fail.

More seriously, a nonexpansive map need not have a fixed point at all. For example,

$$F(x) = x + 1, \qquad x \in \mathbb{R},$$

preserves all distances but has no fixed point.

However, this example uses an unbounded domain. Perhaps boundedness and convexity can force the existence of a fixed point?

Recall that a subset C of a vector space is called *convex* if

$$tx + (1 - t)y \in C$$

whenever $x, y \in C$ and $0 \leq t \leq 1$.

Geometrically, this means that the line segment joining any two points of C lies entirely in C.

In finite dimensions, Brouwer's fixed point theorem gives a very strong answer: every continuous map from a nonempty compact convex subset of $\mathbb{R}^n$ into itself has a fixed point.

In particular, every nonexpansive map of a closed bounded convex subset of $\mathbb{R}^n$ into itself has a fixed point.

But the situation is much more complicated in infinite-dimensional spaces, where closed bounded sets need not be compact.

For example, consider the unit vectors

$$e_1 = (1, 0, 0, \dots), \quad e_2 = (0, 1, 0, \dots), \quad \dots$$

in ℓ^2.

They all belong to the closed unit ball, but

$$\|e_i - e_j\| = \sqrt{2} \qquad (i \neq j).$$

Thus, this sequence has no convergent subsequence, and the unit ball is not compact in the norm topology.

This is where the notion of *reflexivity* enters.

Let X^* denote the space of continuous linear functionals from X to $\mathbb{R}$, and let $X^{**} = (X^*)^*$. Every element $x \in X$ defines an element of X^{**} by evaluation:

$$\ell \longmapsto \ell(x).$$

The Banach space X is called *reflexive* if every element of X^{**} arises in this way.

An important consequence is that every closed bounded convex subset of a reflexive Banach space is *weakly compact*.

The weak topology is defined using continuous linear functionals: a sequence x_n converges weakly to x if

$$\ell(x_n) \longrightarrow \ell(x)$$

for every $\ell \in X^*$.

Thus, reflexivity provides a useful form of compactness, even when ordinary norm compactness fails.

Every Hilbert space, including ℓ^2, is reflexive. So are the familiar spaces L^p for $1 < p < \infty$.

In 1965, Browder (Browder 1965) and Göhde (Göhde 1965) proved that every nonexpansive self-map of a nonempty closed bounded convex subset of a *uniformly convex* Banach space has a fixed point.

Uniform convexity is a geometric condition requiring the unit ball to have no almost-flat portions. More precisely, for every $\varepsilon > 0$ there must exist $\delta > 0$ such that

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \delta$$

whenever $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$.

Every uniformly convex Banach space is reflexive, but not every reflexive Banach space is uniformly convex in its given norm.

In the same year, Kirk (Kirk 1965) established a more general fixed point theorem using the geometric condition of *normal structure*. Roughly speaking, this requires every nontrivial bounded convex subset to contain a point that is not maximally distant from the rest of that subset.

Many subsequent results established the fixed point property for other classes of Banach spaces.

For example, Maurey (Maurey 1981) proved it for reflexive subspaces of L^1, while Domínguez Benavides (Domínguez Benavides 2009) showed that every reflexive Banach space admits an equivalent norm for which the property holds.

But changing the norm can change which maps are nonexpansive. The fundamental question remained whether *every reflexive Banach space has the fixed point property in its original norm*, without additional geometric assumptions.

The new preprint (OpenAI 2026) gives an affirmative answer.

Theorem 1 (OpenAI) *Let X be a real reflexive Banach space, and let $C \subseteq X$ be a nonempty closed bounded convex subset.*
Then every map $F : C \to C$ satisfying

$$\|F(x) - F(y)\| \leq \|x - y\| \qquad \text{for all } x, y \in C$$

has a fixed point. In other words, there exists $x^* \in C$ such that

$$F(x^*) = x^*.$$

The important feature is the complete absence of additional geometric assumptions on the norm.

The space need not be uniformly convex or have normal structure. It need not even be separable, meaning that it need not contain a countable dense subset.

Moreover, the map F is not assumed to be linear, affine, differentiable, or weakly continuous.

The theorem guarantees existence, but not uniqueness or convergence of the successive iterates. For instance, the identity map is nonexpansive and fixes every point.

Let us briefly discuss why reflexivity matters.

Consider the Banach space c_0 of real sequences

$$x = (x_1, x_2, \dots)$$

that converge to zero, equipped with the supremum norm

$$\|x\|_\infty = \sup_j |x_j|.$$

Let C be its closed unit ball, and define

$$F(x_1, x_2, \dots) = (1, x_1, x_2, \dots).$$

The map F takes C into itself and preserves distances:

$$\|F(x) - F(y)\|_\infty = \|x - y\|_\infty.$$

However, a fixed point would have to satisfy

$$x_1 = 1, \quad x_2 = x_1, \quad x_3 = x_2, \quad \dots,$$

so it would have to be the constant sequence $(1, 1, 1, \dots)$. This sequence does not converge to zero and therefore does not belong to c_0.

Thus, F has no fixed point.

The space c_0 is not reflexive, so this example does not contradict the theorem. It illustrates why the infinite-dimensional setting requires more than boundedness, completeness, and convexity.

In fact, Alspach (Alspach 1981) constructed an even subtler counterexample: a nonexpansive map without a fixed point on a weakly compact convex subset of $L^1[0, 1]$. Thus, weak compactness of the domain alone is not sufficient in an arbitrary Banach space.

Let us now discuss the proof.

The authors argue by contradiction, assuming that a reflexive Banach space contains a nonempty closed bounded convex set C and a nonexpansive self-map F without a fixed point.

The first step is to replace C by a smaller set that is *minimal* among the nonempty closed convex subsets preserved by F.

A useful classical observation is that nonexpansive maps always admit *approximate fixed points* on bounded closed convex sets.

Indeed, fix $a \in C$ and an integer $n > 1$. The map

$$G_n(x) = \frac{1}{n}a + \left(1 - \frac{1}{n}\right) F(x)$$

is a strict contraction, with contraction constant $1 - 1/n$.

By Banach's original theorem, G_n has a unique fixed point x_n:

$$x_n = \frac{1}{n}a + \left(1 - \frac{1}{n}\right) F(x_n).$$

Rearranging gives

$$\|F(x_n) - x_n\| = \frac{1}{n}\|F(x_n) - a\| \leq \frac{\operatorname{diam}(C)}{n},$$

where

$$\operatorname{diam}(C) = \sup_{x,y \in C} \|x - y\|.$$

Therefore,

$$\|F(x_n) - x_n\| \longrightarrow 0.$$

The difficulty is that these approximate fixed points need not converge in norm. Even though reflexivity supplies weakly convergent subsequences, a nonexpansive map need not preserve their weak limits.

The proof uses a classical theorem of Goebel and Karlovitz (Goebel 1975; Karlovitz 1976): in a minimal invariant set with no fixed point, every approximate fixed point sequence eventually becomes almost maximally distant from every fixed point of the set.

The authors combine this property with a new construction involving carefully chosen nonexpansive maps, convex combinations, and an infinite tree of points and linear functionals.

At each level of the tree, the functionals detect certain large distances between approximate fixed points. At the same time, the construction arranges for these functionals to become small on every fixed vector in a suitable subspace.

A delicate argument involving weighted changes of the tree's terminal points then produces a contradiction. Weak compactness ensures that the finite configurations can be combined into one limiting configuration, where a vector is simultaneously detected by infinitely many functionals that should converge to zero on that vector.

This rules out the assumed fixed-point-free map.

The proof is technically sophisticated, but its starting point is very natural: replace a nonexpansive map by a sequence of genuine contractions, study their approximate fixed points, and exploit the geometry of a reflexive space to show that some exact fixed point must exist.

The theorem has a further consequence concerning several transformations.

A family of maps is called *commuting* if

$$F(G(x)) = G(F(x))$$

for every pair of maps F, G in the family.

Combining the new theorem with Bruck's common fixed point theorem (Bruck 1974), the authors deduce that every pairwise commuting family of nonexpansive self-maps of a nonempty closed bounded convex subset of a real reflexive Banach space has a *common fixed point*.

Thus, there exists one point fixed simultaneously by every map in the family, even if the family is infinite.

The paper is currently available as a preprint, and independent mathematical scrutiny remains important.

If its proof is confirmed, it will complete a major chapter in fixed point theory that began with the work of Browder, Göhde, and Kirk in 1965.

The conclusion is strikingly simple: in every real reflexive Banach space, the geometry of closed bounded convex sets is sufficient to force fixed points for all nonexpansive self-maps, even when no strict contraction is present.

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