Fourier Restriction and the Kakeya Conjecture

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In September and October 2026, OpenAI released two preprints (OpenAI 2026a, 2026b) announcing proofs of major conjectures in harmonic analysis. The first establishes the sharp range of Fourier restriction estimates for positively curved surfaces in three dimensions. The second extends these results to spheres and paraboloids in all dimensions, and, as a consequence, establishes the Kakeya conjecture in every dimension.

The two theorems are closely connected, so we discuss them together.

Let us begin with the definitions.

For an integrable function $F : \mathbb{R}^n \to \mathbb{C}$, its *Fourier transform* is defined by

$$\widehat{F}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} F(x)\, dx,$$

where $x \cdot \xi$ denotes the usual scalar product.

The Fourier transform is one of the fundamental tools of mathematical analysis. Roughly speaking, it decomposes a function into waves of different frequencies.

A natural question is whether the Fourier transform of a function can be meaningfully *restricted* to a lower-dimensional surface.

For example, consider the unit sphere

$$S^{n-1} = \{\xi \in \mathbb{R}^n : |\xi| = 1\}.$$

This sphere has zero n-dimensional Lebesgue measure, so knowing a function almost everywhere in $\mathbb{R}^n$ does not tell us its values on the sphere.

Nevertheless, it may be possible to control the Fourier transform on the sphere using an integral norm of the original function.

Such questions are known as *Fourier restriction problems*.

It is often more convenient to study the corresponding *extension operator*. Given a function $f : S^{n-1} \to \mathbb{C}$, define

$$E_{S^{n-1}} f(x) = \int_{S^{n-1}} e^{ix\cdot\xi} f(\xi)\, d\sigma(\xi),$$

where $d\sigma$ is the usual surface measure on the sphere.

The connection with restriction comes from an integral identity: the extension operator is, up to complex conjugation, the adjoint of the operation restricting a Fourier transform to the sphere.

The problem is to determine which inequalities of the form

$$\|E_{S^{n-1}} f\|_{L^b(\mathbb{R}^n)} \le C \|f\|_{L^a(S^{n-1})}$$

can hold.

Here, for $1 \le p < \infty$,

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p \right)^{1/p},$$

with the appropriate measure on Ω, while the L^∞ norm is the essential supremum of $|f|$.

A closely related problem concerns the *paraboloid*. Let B^{n-1} denote the unit ball in $\mathbb{R}^{n-1}$. The corresponding extension operator is

$$E_{\mathrm{par}} f(x', x_n) = \int_{B^{n-1}} e^{i(x'\cdot\xi + x_n|\xi|^2)} f(\xi)\, d\xi,$$

where $x' \in \mathbb{R}^{n-1}$.

In 1979, Stein (Stein 1979) formulated the conjecture that

$$\|E_{\mathrm{par}} f\|_{L^p(\mathbb{R}^n)} \le C_p \|f\|_{L^p(B^{n-1})}$$

whenever

$$p > \frac{2n}{n-1}.$$

The same critical exponent appears in restriction problems for spheres and other curved surfaces.

Why this particular number?

Consider the case $n = 3$ and take the constant function $f = 1$ on the unit sphere S^2.

A direct calculation gives

$$E_{S^2} 1(x) = 4\pi \frac{\sin|x|}{|x|}.$$

Thus, for large $|x|$, the absolute value of this function is typically of order $1/|x|$.

To have $E_{S^2}1 \in L^p(\mathbb{R}^3)$, we would need the integral

$$\int_1^\infty r^{2-p} |\sin r|^p \, dr$$

to converge.

This happens precisely when $p > 3$.

Consequently, an inequality

$$\|E_{S^2} f\|_{L^p(\mathbb{R}^3)} \le C_p \|f\|_{L^p(S^2)}$$

cannot hold for $p \le 3$.

In higher dimensions, the Fourier extension of a constant function on the sphere has a similar oscillatory decay, with size of order

$$|x|^{-(n-1)/2}.$$

This explains the critical exponent

$$p = \frac{2n}{n-1}.$$

The restriction conjecture predicts that the obvious obstruction is essentially the only one: the inequality should hold for every strictly larger p.

Proving this turned out to be extraordinarily difficult.

A major advance came from the *polynomial partitioning method*, developed by Guth and Katz in incidence geometry and subsequently applied by Guth (Guth 2016) to Fourier restriction.

In three dimensions, Guth proved the conjectured type of estimate for $p > 13/4$ with bounded input data. Further work improved the exponent, and in 2024 Wang and Wu (Wang and Wu 2024) established the diagonal estimate for

$$p > \frac{22}{7}.$$

Since $22/7 > 3$, the conjectured range was still not completely covered.

The first new OpenAI preprint (OpenAI 2026a), dated September 24, 2026, claims to close this gap.

Theorem 1 (OpenAI) *Let $\Sigma \subset \mathbb{R}^3$ be a compact smooth surface with strictly positive curvature, possibly with smooth boundary. Define*

$$E_\Sigma f(x) = \int_\Sigma e^{ix\cdot\xi} f(\xi)\, d\sigma(\xi).$$

Then, for every $p > 3$, there exists a constant $C_{\Sigma,p}$ such that

$$\|E_\Sigma f\|_{L^p(\mathbb{R}^3)} \le C_{\Sigma,p} \|f\|_{L^p(\Sigma)}$$

for every $f \in L^p(\Sigma)$.

The theorem reaches the optimal strict range $p > 3$, not only for spheres and paraboloids but for every compact smooth positively curved surface in $\mathbb{R}^3$.

Here positive curvature means that, locally, the surface bends in the same direction along every tangent direction, as happens for a sphere.

The result requires no symmetry of the surface. In particular, it applies to smooth surfaces that are very different from a sphere.

The second preprint (OpenAI 2026b), released on October 25, 2026, makes a further breakthrough. Its principal result concerns the *local smoothing* of solutions to the wave equation, but it also establishes Fourier extension estimates in all dimensions.

Moreover, it proves not just the diagonal L^p estimate, but a more general family of inequalities involving different exponents.

For $1 < a \le \infty$, let a' denote its *conjugate exponent*, defined by

$$\frac{1}{a} + \frac{1}{a'} = 1.$$

In particular, $a' = 1$ when $a = \infty$.

The new result is the following.

Theorem 2 (OpenAI) *Let $n \ge 2$ and $1 < a \le \infty$. Suppose that $b < \infty$ satisfies*

$$b > \frac{2n}{n-1}, \qquad b > \frac{n+1}{n-1}a'.$$

Then there exists a constant $C = C(n, a, b)$ such that

$$\|E_{S^{n-1}} f\|_{L^b(\mathbb{R}^n)} \le C \|f\|_{L^a(S^{n-1})}$$

for every $f \in L^a(S^{n-1})$.
The same conclusion holds for the compact paraboloid:

$$\|E_{\mathrm{par}} g\|_{L^b(\mathbb{R}^n)} \le C \|g\|_{L^a(B^{n-1})}$$

for every $g \in L^a(B^{n-1})$.
Taking $a = b = p$, the two inequalities on the exponents reduce to the single condition

$$p > \frac{2n}{n-1}.$$

Thus, the theorem proves Stein’s conjecture for the paraboloid in every dimension.

For example, the critical exponents are

n	Required exponent
2	$p > 4$
3	$p > 3$
4	$p > 8/3$
5	$p > 5/2$

The bounds are sharp in the sense that the stated strict ranges cannot be enlarged in general. The theorem does not claim boundedness at the critical endpoints.

The general theorem applies to spheres and compact paraboloids. The earlier three-dimensional theorem remains stronger in one respect: it applies to arbitrary compact smooth positively curved surfaces, not only these standard models.

Let us briefly discuss the proof ideas.

The first difficulty is that Fourier extension involves many oscillating waves that can interfere constructively or destructively.

A useful technique is to divide a curved surface into small pieces, called *frequency caps*. The extension of a function supported on one small cap is concentrated, at appropriate scales, in a long thin region of physical space called a *wave packet*.

The central problem is therefore to understand how many such wave packets can overlap and how their oscillations interact.

The three-dimensional proof combines estimates for the propagation of wave packets with geometric bounds on concentrations of thin tubes. It also uses a technique known as *epsilon removal*, which allows certain estimates with small losses on large balls to be converted into global estimates for slightly larger exponents.

The all-dimensional proof follows a different route, starting from the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u.$$

The key idea is that the evolution of a wave can become highly concentrated when many incoming rays meet near a focus.

The authors establish sharp estimates for the average size of such waves over a time interval. These are called *local smoothing estimates*.

By constructing waves whose amplitudes depend on their directions and examining them near a focus, they recover the Fourier extension operator on the sphere.

The resulting estimates lead to the full conjectured range of Fourier restriction inequalities.

But perhaps the most remarkable consequence concerns the *Kakeya conjecture*.

A *Kakeya set* in $\mathbb{R}^n$ is a set containing a line segment of length 1 in every possible direction.

Such a set sounds as though it should occupy a considerable amount of space. Surprisingly, Besicovitch showed that there exist Kakeya sets in the plane with Lebesgue measure zero.

Thus, a set can contain a unit line segment in every direction and still have zero area.

However, Besicovitch's construction is extremely complicated, and the resulting set has full *Hausdorff dimension*.

Hausdorff dimension is a notion of size that also applies to fractal sets. An ordinary smooth curve has Hausdorff dimension 1, a two-dimensional region has dimension 2, and a complicated fractal may have a noninteger dimension.

In 1971, Davies proved that every planar Kakeya set has Hausdorff dimension 2.

The *Kakeya conjecture* predicted that the same phenomenon holds in every dimension:

$$\dim_H(K) = n$$

for every Kakeya set $K \subseteq \mathbb{R}^n$.

This remained open for decades.

In 2025, Hong Wang and Joshua Zahl (Wang and Zahl 2025) proved the conjecture in dimension three, as discussed in my earlier blog post.

The new Fourier restriction theorem establishes the conjecture in *every dimension*.

The connection between the two problems comes from the wave packets described above.

A small patch of frequencies produces a wave concentrated in a thin tube. Different patches produce tubes pointing in different directions.

If many such tubes were allowed to overlap too efficiently, the resulting superposition of waves could violate the Fourier extension estimates.

This connection was developed by Bourgain (Bourgain 1991) and other mathematicians. The new theorem gives sufficiently strong extension estimates to control the overlap of tubes in every dimension.

In fact, the authors prove a quantitative form of the *Kakeya maximal conjecture*.

For a function $F : \mathbb{R}^n \to \mathbb{C}$, a direction $\omega \in S^{n-1}$, and a number $0 < \delta < 1$, consider all tubes of length 1 and radius δ pointing in direction ω. Define

$$K_\delta F(\omega) = \sup_{T} \frac{1}{|T|} \int_T |F(x)|\, dx,$$

where the supremum is taken over all such tubes T, and $|T|$ denotes their volume.

The theorem implies that, for every $\varepsilon > 0$,

$$\|K_\delta F\|_{L^n(S^{n-1})} \le C_{n,\varepsilon} \delta^{-\varepsilon} \|F\|_{L^n(\mathbb{R}^n)}.$$

In fact, the proof obtains an even stronger estimate in which the factor $\delta^{-\varepsilon}$ is replaced by a fixed power of $\log(2/\delta)$.

This says that the average concentration of a function along thin tubes in all directions cannot be arbitrarily large.

A classical covering argument then gives the full Kakeya conjecture:

$$\boxed{\dim_H(K) = n \quad \text{for every Kakeya set } K \subseteq \mathbb{R}^n.}$$

Thus, even though a Kakeya set may have zero Lebesgue measure, it must have the largest possible Hausdorff dimension.

The conclusion holds in every dimension $n \ge 2$, including dimensions in which the Kakeya conjecture had previously been open.

The two new papers are currently available as preprints, and their proofs await independent mathematical verification. If confirmed, they will settle several central conjectures in harmonic analysis simultaneously.

What makes the result especially satisfying is the connection between two apparently different questions. One asks how oscillating waves can combine when their frequencies lie on a curved surface. The other asks how efficiently thin tubes pointing in every direction can overlap.

The new theorems show that both questions are governed by the same underlying geometric restrictions. Sharp control of Fourier extension forces Kakeya sets to have full dimension, completing a story that began with Besicovitch’s surprising constructions more than a century ago.

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