The Hot Spots Conjecture for Smooth Simply Connected Domains

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In January 2026, I wrote a blog post about the hottest and coldest points on a triangle. In March, I discussed the hot spots conjecture for lip domains. Both posts concerned a famous problem in spectral geometry: *if heat diffuses through an insulated plate, must its hottest and coldest points eventually lie on the boundary?*

In September 2026, OpenAI released a preprint (OpenAI 2026) announcing a proof of this conjecture for every smooth simply connected planar domain. Remarkably, the result is considerably stronger: it rules out *all interior critical points*, not merely interior maxima and minima.

Let us recall the problem.

Imagine a thin metal plate occupying a bounded domain $\Omega \subset \mathbb{R}^2$. We assume that the plate is insulated along its boundary, so that heat cannot enter or leave.

If $v(x,t)$ denotes the temperature at the point $x \in \Omega$ and time t, its evolution is described by the *heat equation*

$$\frac{\partial v}{\partial t} = \Delta v,$$

where

$$\Delta v = \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2}$$

is the *Laplacian*.

Insulation is expressed by the *Neumann boundary condition*

$$\frac{\partial v}{\partial n} = 0 \qquad \text{on } \partial\Omega,$$

where n is the outward unit normal to the boundary.

As time passes, the temperature approaches its average value over the plate. However, the spatial pattern of the remaining temperature differences is governed by the eigenfunctions of the Laplacian.

More precisely, consider the equation

$$\Delta u = -\mu u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

A number μ for which this equation has a nonzero solution is called a *Neumann eigenvalue*, and the corresponding function u is called a *Neumann eigenfunction*.

The smallest eigenvalue is $\mu_1 = 0$, corresponding to constant functions. The next eigenvalue, denoted by $\mu_2 > 0$, is called the *second Neumann eigenvalue* or the *first positive Neumann eigenvalue*.

For a typical initial temperature distribution, the leading contribution to the deviation from the average temperature, at sufficiently large times, is proportional to

$$e^{-\mu_2 t} u(x),$$

where u is a second Neumann eigenfunction. If μ_2 has multiplicity greater than one, u may be a linear combination of several eigenfunctions.

Thus, understanding the maximum and minimum points of u helps us understand where the plate remains relatively hot or cold for the longest time.

In 1975, Jeffrey Rauch (Rauch 1975) proposed the *hot spots conjecture*, predicting that the maximum and minimum of a second Neumann eigenfunction should occur on the boundary of the domain.

The conjecture is easy to verify in some simple examples.

For instance, on the one-dimensional interval $(0, L)$, the first nonconstant Neumann eigenfunction is

$$u(x) = \cos\left(\frac{\pi x}{L}\right).$$

Its maximum is attained at $x = 0$ and its minimum at $x = L$.

Moreover,

$$u'(x) = -\frac{\pi}{L} \sin\left(\frac{\pi x}{L}\right)$$

is nonzero for every $x \in (0, L)$. Thus, the function has no critical points in the interior.

But in two dimensions, where the shape of the domain can be much more complicated, the problem is substantially harder.

In fact, the conjecture is false for general planar domains. In 1999, Burdzy and Werner (Burdzy and Werner 1999) constructed a counterexample involving a domain with holes.

This suggested that the topology of the domain matters. A natural restriction is to consider *simply connected domains*, meaning, roughly speaking, domains without holes: every closed curve in the domain can be continuously contracted to a point while remaining inside it.

Important partial results were obtained for special classes of such domains.

In 2004, Atar and Burdzy (Atar and Burdzy 2004) proved the conjecture for a broad class of domains called *lip domains*. Their proof used probabilistic techniques involving reflected Brownian motion.

Another major advance came in 2020, when Judge and Mondal (Judge and Mondal 2020, 2022) proved the conjecture for all triangles. More recently, Chen, Gui, and Yao (Chen et al. 2026) obtained a finer classification of the possible critical points of second Neumann eigenfunctions on triangles.

However, these results did not settle the conjecture for arbitrary simply connected planar domains.

Even if a domain has no holes, it may be highly nonconvex, with thin necks, long winding corridors, and complicated boundary geometry.

The new preprint (OpenAI 2026) claims to prove the conjecture for all such domains, provided their boundaries are smooth.

Theorem 1 (OpenAI) *Let $\Omega \subset \mathbb{R}^2$ be a nonempty bounded simply connected domain with infinitely differentiable boundary.*
Let u be any nonzero real eigenfunction corresponding to the first positive Neumann eigenvalue $\mu_2(\Omega)$.
Then

$$\nabla u(x) \neq 0 \qquad \text{for every } x \in \Omega.$$

Consequently, for every $x \in \Omega$,

$$\min_{y \in \partial\Omega} u(y) < u(x) < \max_{y \in \partial\Omega} u(y).$$

The second conclusion says that every interior point is strictly warmer than the coldest boundary point and strictly colder than the hottest boundary point, in the temperature pattern described by u.

But the first conclusion is even more remarkable.

Recall that a *critical point* of a differentiable function u is a point where its gradient vanishes:

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \right) = 0.$$

Every interior maximum or minimum is a critical point. However, critical points can also be *saddle points*, where the function increases in some directions and decreases in others.

For example,

$$u(x, y) = x^2 - y^2$$

has a critical point at the origin, although the origin is neither a local maximum nor a local minimum.

Thus, the hot spots conjecture only excludes certain critical points, whereas the new theorem excludes *every* critical point in the interior.

This stronger property was not part of Rauch's original conjecture.

Another important feature is that the theorem applies to *every* eigenfunction corresponding to μ_2, even if this eigenvalue has multiplicity greater than one.

In particular, it does not merely guarantee the existence of one eigenfunction with good behaviour. Every nonzero linear combination of eigenfunctions belonging to this eigenspace has no interior critical points.

Let us briefly discuss the proof.

The authors start from a variational principle developed by Rohleder (Rohleder 2024), involving vector fields rather than scalar functions.

Consider the gradient

$$X = \nabla u.$$

Because u satisfies the Neumann boundary condition, X is tangent to the boundary of Ω.

The key observation is that gradients of second Neumann eigenfunctions are precisely the vector fields for which a certain inequality involving their divergence and curl becomes an equality.

For a suitable tangent vector field X, this inequality has the form

$$\int_\Omega \left((\operatorname{div} X)^2 + (\operatorname{curl} X)^2 \right) \, dx \geq \mu_2 \int_\Omega |X|^2 \, dx.$$

Here

$$\operatorname{div} X = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2}, \qquad \operatorname{curl} X = \frac{\partial X_2}{\partial x_1} - \frac{\partial X_1}{\partial x_2}.$$

The authors use a conformal map to transform the original domain into the unit disk, where conformal maps are holomorphic bijections preserving angles.

They then construct special functions on the boundary of the disk using Green functions and their associated kernels.

These functions act as multipliers of the boundary values of ∇u. The remarkable feature of the construction is that, if ∇u vanished at an interior point, the variational inequality would force several of these modified boundary fields to arise from other second Neumann eigenfunctions.

The final step uses a classical restriction on the dimension of the second Neumann eigenspace.

For a smooth simply connected planar domain, this eigenspace has dimension at most two.

The authors show that the existence of an interior critical point would either produce three linearly independent eigenfunctions in this space, contradicting the dimension bound, or violate a boundary uniqueness principle for solutions of the eigenvalue equation.

Both possibilities lead to a contradiction.

The main technical difficulty lies in constructing the boundary multipliers and establishing the necessary positivity properties of their kernels. The proof combines spectral theory, conformal mapping, and Hilbert-space geometry.

It is worth emphasizing two limitations of the theorem.

First, the assumption of simple connectivity is essential for this general conclusion: counterexamples exist for domains with holes.

Second, the boundary is assumed to be infinitely differentiable. Thus, the theorem does not directly apply to triangles or other polygonal domains, which have corners. Those require separate arguments, such as the results discussed in my earlier posts.

Since the preprint appeared, there have already been important extensions. In October 2026, Jizhou Guo (Guo 2026) proved that *every* bounded convex planar domain, even with a nonsmooth boundary, satisfies the original hot spots conjecture: every nonzero second Neumann eigenfunction attains both its maximum and its minimum on the boundary. No assumption on the simplicity of the eigenvalue is needed, so the result also applies when μ_2 is multiple. Guo's proof deduces this conclusion from the OpenAI theorem by approximation; it does not establish the stronger absence of all interior critical points. Independently, Deng, Jiang, and Yang (Deng et al. 2026) proved that the maximum of the *absolute value* of every such eigenfunction is attained only on

the boundary of any bounded convex planar domain, again without smoothness or simplicity assumptions. Their result gives a strict maximum-modulus inequality, although it does not separately assert strict inequalities for both the maximum and the minimum.

Taken together, these results substantially extend the hot spots conjecture beyond smooth domains. In particular, the original boundary-extrema conclusion now covers every bounded convex planar domain, including polygons and domains with corners. However, the general case of arbitrary nonsmooth, nonconvex simply connected planar domains is not yet settled without additional assumptions.

The original OpenAI paper and these extensions are recent preprints, and independent scrutiny of the proofs remains important.

If confirmed, they provide a remarkably strong description of the first nonconstant mode of heat diffusion on an insulated planar plate without holes.

For a smooth plate, no matter how complicated its shape is, the corresponding eigenfunction has no stationary points whatsoever in its interior. For an arbitrary bounded convex plate, even if its boundary has corners, both its hottest and coldest points occur on the boundary.

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