Fourier Series at the $L \log L$ Threshold

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In September 2026, OpenAI released a preprint (OpenAI 2026) announcing a proof of the famous $L \log L$ conjecture on the almost-everywhere convergence of Fourier series. The result strengthens Carleson's celebrated theorem of 1966 and resolves a problem that had remained open for decades.

The question is deceptively simple: *if we represent a periodic function by its Fourier series, under what conditions does the series actually converge to the original function?*

Let us recall the definitions.

Consider a complex-valued function $f : \mathbb{R} \to \mathbb{C}$ with period 1, so that

$$f(x+1) = f(x)$$

for every real number x.

Its *Fourier coefficients* are defined by

$$\widehat{f}(k) = \int_0^1 f(x) e^{-2\pi i k x}\, dx, \qquad k \in \mathbb{Z},$$

provided that f is integrable, meaning that

$$\int_0^1 |f(x)|\, dx < \infty.$$

Here $\mathbb{Z}$ denotes the set of integers and $i^2 = -1$.

The corresponding *Fourier series* is

$$\sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{2\pi i k x}.$$

To make sense of convergence, we consider the *symmetric partial sums*

$$S_N f(x) = \sum_{k=-N}^{N} \widehat{f}(k) e^{2\pi i k x}, \qquad N = 0, 1, 2, \dots.$$

The fundamental question is whether

$$\lim_{N \to \infty} S_N f(x) = f(x).$$

For example, if

$$f(x) = 2 + 3\cos(2\pi x) - \sin(4\pi x),$$

then f is itself a finite Fourier sum, so $S_N f(x) = f(x)$ for every x and every $N \geq 2$.

However, not every periodic function can be represented by a finite Fourier sum, and convergence of infinite Fourier series is much more complicated.

Even for very simple functions, convergence need not hold at every point. For example, consider the function

$$f(x) = \begin{cases} 1, & 0 \leq x < 1/2, \\ 0, & 1/2 \leq x < 1, \end{cases}$$

extended periodically.

At the points of discontinuity, its Fourier series converges to the average of the two one-sided limits, namely $1/2$, rather than to the chosen value of f.

Thus, it is natural to ask for convergence *almost everywhere*, meaning at every point except possibly a set of Lebesgue measure zero.

A major breakthrough was obtained in 1966, when Lennart Carleson (Carleson 1966) proved that the Fourier series of every square-integrable function converges almost everywhere.

More precisely, Carleson’s theorem states that if

$$\int_0^1 |f(x)|^2 \, dx < \infty,$$

then

$$S_N f(x) \longrightarrow f(x)$$

for almost every x.

This settled a famous problem posed by Lusin in 1915.

In 1968, Richard Hunt (Hunt 1968) extended the theorem to every space L^p with $p > 1$. Here $f \in L^p$ means that

$$\int_0^1 |f(x)|^p \, dx < \infty.$$

Thus, for every fixed $p > 1$, the Fourier series of every function in L^p converges almost everywhere.

But what happens when $p = 1$?

Unfortunately, the analogous statement is false. In 1923, Kolmogorov (Kolmogoroff 1923) constructed an integrable function whose Fourier series diverges almost everywhere.

Therefore, the condition $f \in L^1$ is not sufficient.

This leaves an intriguing gap. Carleson and Hunt tell us that convergence holds for every L^p, $p > 1$, while Kolmogorov's example shows that it fails in general for L^1.

Can we find a natural class of functions lying between these two extremes for which convergence is still guaranteed?

One of the most important candidates is the class $L \log L$.

A function f belongs to $L \log L$ if

$$\int_0^1 |f(x)| \log(2 + |f(x)|)\, dx < \infty,$$

where $\log$ denotes the natural logarithm.

This condition is stronger than ordinary integrability because of the extra logarithmic factor, but it is much weaker than requiring integrability of $|f|^p$ for any fixed $p > 1$.

To see the difference, consider the function

$$f(x) = \frac{1}{x(\log(1/x))^3}, \qquad 0 < x < e^{-2},$$

and put $f(x) = 0$ for $e^{-2} \le x < 1$, extending it periodically.

This function is unbounded near $x = 0$. Nevertheless, a direct calculation shows that

$$\int_0^1 |f(x)| \log(2 + |f(x)|)\, dx < \infty.$$

Indeed, near zero the integrand behaves like

$$\frac{1}{x(\log(1/x))^2},$$

whose integral converges.

On the other hand,

$$\int_0^1 |f(x)|^p\, dx = \infty \qquad \text{for every } p > 1.$$

Thus, this function belongs to $L \log L$ but to none of the spaces L^p, $p > 1$.

Carleson's and Hunt's theorems do not establish almost-everywhere Fourier convergence for this function.

Could convergence nevertheless hold for every function in $L \log L$?

This became known as the $L \log L$ *Fourier-convergence conjecture*.

Several important partial results were obtained. In particular, Sjölin established convergence under an integrability condition involving additional logarithmic factors, and Antonov (Antonov 1996) subsequently weakened the requirement to a class commonly denoted by

$$L \log L \log\log\log L.$$

This means, roughly speaking, that the integrability condition involves $|f|$ multiplied by $\log |f|$ and three further iterated logarithms. The logarithms are suitably modified near zero to avoid difficulties with their definitions.

Later, Arias de Reyna, Lie (Lie 2017), and other mathematicians developed further improvements using sophisticated techniques from harmonic analysis.

However, the additional factors could not be completely removed.

The new OpenAI preprint (OpenAI 2026) claims to establish the desired result at the exact $L \log L$ level.

Theorem 1 (OpenAI) *Let $f : \mathbb{R} \to \mathbb{C}$ be a measurable function with period 1 satisfying*

$$\int_0^1 |f(x)| \log(2 + |f(x)|)\, dx < \infty.$$

Then its symmetric Fourier partial sums

$$S_N f(x) = \sum_{k=-N}^{N} \widehat{f}(k) e^{2\pi i k x}$$

satisfy

$$\lim_{N \to \infty} S_N f(x) = f(x)$$

for almost every $x \in [0, 1]$.
The theorem substantially extends the classical Carleson–Hunt result.

In particular, it guarantees almost-everywhere convergence for the unbounded function constructed above, even though this function does not belong to L^p for any $p > 1$.

There are two important points to emphasize.

First, the theorem concerns convergence along the *entire sequence*

$$S_0 f(x), S_1 f(x), S_2 f(x), \ldots,$$

not merely along a specially chosen subsequence.

Second, it concerns the ordinary Fourier partial sums, without any averaging.

This distinction matters because the *Cesàro averages*

$$\frac{1}{N+1}\sum_{j=0}^{N} S_j f(x)$$

were already known to converge almost everywhere to f for every integrable function. Averaging smooths out some of the oscillations responsible for divergence, so the corresponding theorem is much easier.

The new result requires no such smoothing.

Let us briefly discuss the proof.

A central tool is the *Carleson maximal operator*, defined by

$$Cf(x) = \sup_{N \geq 0} |S_N f(x)|.$$

Instead of studying the convergence of each individual Fourier partial sum separately, one tries to control the largest possible partial sum at each point.

This is a standard strategy in harmonic analysis: if a suitable estimate for the maximal operator is available, one can first prove convergence for simple functions, such as trigonometric polynomials, and then extend it to more general functions by approximation.

However, obtaining the necessary maximal estimate near L^1 is extremely difficult.

The proof begins by decomposing the function into pieces supported on intervals of different lengths. These intervals form a binary tree, with each interval divided into two smaller intervals.

At every stage, the authors compare averages over the two halves and analyze how they interact with the oscillating factors

$$e^{2\pi i k x}$$

appearing in Fourier analysis.

The difficulty is that there are many possible frequencies k, and estimating them independently would produce an error that is too large.

The authors develop a method using probability and information-theoretic estimates to identify groups of frequencies whose oscillations behave similarly. This permits them to control many frequencies simultaneously.

Another important ingredient is a representation of the *Hilbert transform* through averages of certain operators associated with dyadic intervals. This connects the combinatorial estimates on the binary tree with the ordinary Fourier partial sums.

The argument first establishes a strong estimate outside a carefully controlled exceptional set. The function is then separated into pieces according to the sizes of its values.

The final step combines these estimates over all pieces. The total cost is controlled precisely by

$$\int_0^1 |f(x)| \log(2 + |f(x)|)\, dx,$$

which explains why the $L \log L$ condition appears naturally.

The resulting estimate is strong enough to prove almost-everywhere convergence of the original Fourier series.

The theorem does not imply that Fourier series converge almost everywhere for every integrable function: Kolmogorov's counterexample still applies. Nor does it assert that $L \log L$ is the largest possible class for which such a convergence theorem can hold.

Rather, it settles a particularly natural and longstanding question about how far Carleson's theorem can be extended towards L^1.

The preprint is accompanied by a Lean formalization of its main statement, although independent mathematical scrutiny of the result remains valuable.

The conclusion is striking. The Fourier series of a function may involve infinitely many oscillating terms, and the function itself may have very large values near some points. Nevertheless, provided that the function satisfies an integrability condition only slightly stronger than L^1, the ordinary Fourier partial sums still recover the function almost everywhere.

References

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