De Giorgi’s Conjecture: The Sharp Dimension

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https://functor.network/user/3333/entry/2072

In June 2026, I wrote a blog post about De Giorgi’s conjecture, one of the most beautiful problems in the theory of partial differential equations. The post focused on a theorem of Ovidiu Savin (Savin 2009), who proved the conjecture in dimensions up to eight under an additional limiting assumption.

I updated that post in September, after Liu, Luo, Wang, J. Wei, Y. Wei and Wu (Liu et al. 2026) announced a proof of the full conjecture in dimension four.

On September 26, 2026, OpenAI released a preprint (OpenAI 2026) that goes considerably further, announcing a proof of the conjecture in every dimension up to eight, without any additional assumptions.

Let us briefly recall the problem.

The *Allen–Cahn equation* is

$$\Delta u = u^3 - u,$$

where $u : \mathbb{R}^n \to \mathbb{R}$ and

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}$$

is the *Laplacian* of u.

This equation arises in mathematical models of phase transitions. One should imagine a material that can exist in two preferred states, represented by the values -1 and 1, with a transition region separating them.

The simplest nonconstant solution is

$$g(t) = \tanh\left(\frac{t}{\sqrt{2}}\right).$$

Indeed, direct differentiation shows that

$$g'' = g^3 - g.$$

Moreover,

$$g'(t) > 0, \qquad \lim_{t\to-\infty} g(t) = -1, \qquad \lim_{t\to+\infty} g(t) = 1.$$

Consequently, for any unit vector $e \in \mathbb{R}^n$ and any constant $c \in \mathbb{R}$, the function

$$u(x) = g(e \cdot x - c) = \tanh\left(\frac{e \cdot x - c}{\sqrt{2}}\right)$$

solves the Allen–Cahn equation.

Here $e \cdot x$ denotes the usual scalar product. The function depends on only one linear coordinate, so its level sets

$$\{x \in \mathbb{R}^n : u(x) = a\}$$

are hyperplanes.

In 1978, Ennio De Giorgi (De Giorgi 1979) conjectured that, in dimensions $n \leq 8$, these are essentially the only solutions satisfying two natural conditions:

$$|u(x)| < 1 \qquad \text{and} \qquad \frac{\partial u}{\partial x_n}(x) > 0 \quad \text{for every } x \in \mathbb{R}^n.$$

The second condition says that u is strictly increasing in the last coordinate. Thus, the transition between the two phases always proceeds in the same direction.

But does this monotonicity force the transition layer to be completely flat?

The answer was known to be affirmative in dimensions two and three, by results of Ghoussoub and Gui in 1998 and Ambrosio and Cabré in 2000.

In 2009, Savin (Savin 2009) extended this conclusion to all dimensions $n \leq 8$, provided that one also assumes

$$\lim_{t \to -\infty} u(x', t) = -1, \qquad \lim_{t \to +\infty} u(x', t) = 1$$

for every fixed $x' \in \mathbb{R}^{n-1}$.

This additional assumption guarantees that along every vertical line, the solution makes a complete transition from one phase to the other.

It looks natural, but it does not follow immediately from monotonicity. A bounded increasing function can approach limits strictly between -1 and 1.

Thus, Savin’s theorem left open the original version of De Giorgi’s conjecture.

The new preprint (OpenAI 2026) announces the following result.

Theorem 1 (OpenAI) *Let $1 \le n \le 8$, and suppose that $u : \mathbb{R}^n \to (-1,1)$ is a twice continuously differentiable solution of*

$$\Delta u = u^3 - u$$

such that

$$\frac{\partial u}{\partial x_n} > 0 \qquad \text{everywhere in } \mathbb{R}^n.$$

Then there exist a unit vector $e \in \mathbb{R}^n$ with $e_n > 0$ and a constant $c \in \mathbb{R}$ such that

$$u(x) = \tanh\left(\frac{e \cdot x - c}{\sqrt{2}}\right).$$

In particular, every nonempty level set of u is a hyperplane.

The theorem completely removes Savin's additional limiting assumption. It also does not require any restriction on the growth of the energy of the solution.

Thus, in dimensions up to eight, *monotonicity alone, together with boundedness and the Allen–Cahn equation, forces the solution to be one-dimensional*.

Why is the number eight so important?

As discussed in my earlier post, De Giorgi's conjecture is closely connected with the famous Bernstein problem about minimal surfaces.

A *minimal hypersurface* is, roughly speaking, a surface that is stationary for the area functional under small deformations. The level sets of Allen–Cahn solutions can behave like minimal hypersurfaces when viewed at large spatial scales.

In low dimensions, minimal graphs must be hyperplanes. However, Bombieri, De Giorgi and Giusti showed in 1969 that nonplanar entire minimal graphs exist in ambient dimensions nine and higher.

Using such minimal graphs, del Pino, Kowalczyk and Wei (Pino et al. 2011) constructed counterexamples to De Giorgi's conjecture in every dimension $n \ge 9$.

Their solutions satisfy the Allen–Cahn equation and are strictly increasing in one direction, but their level sets are not hyperplanes.

Therefore, the 2026 theorem establishes the *exact dimensional threshold*:

$$\begin{aligned} n \le 8 &: \quad \text{every bounded strictly monotone solution is planar,} \\ n \ge 9 &: \quad \text{nonplanar strictly monotone solutions exist.} \end{aligned}$$

Let us briefly discuss the proof.

One of the most interesting aspects is that the authors first establish a stronger theorem about *stable solutions* in dimension seven.

The Allen–Cahn equation is associated with the energy

$$E(u;\Omega) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2 \right) \, dx.$$

A solution is called *stable* if every sufficiently small perturbation supported in a bounded region has nonnegative second variation of this energy.

Stability is a weaker condition than minimizing the energy: a stable solution need not be a global minimizer.

The authors prove that every stable solution

$$v : \mathbb{R}^7 \to [-1,1]$$

of the Allen–Cahn equation is either the constant function 1, the constant function -1, or a one-dimensional transition

$$v(x) = \tanh\left(\frac{a \cdot x - b}{\sqrt{2}} \right).$$

Importantly, this classification requires *no assumption on energy growth*.

This is itself a major rigidity theorem.

How does it help with the original eight-dimensional conjecture?

Suppose that u is a bounded solution in $\mathbb{R}^8$ that is strictly increasing in the last coordinate.

The function

$$h = \frac{\partial u}{\partial x_8}$$

is positive everywhere. Differentiating the Allen–Cahn equation shows that

$$-\Delta h + (3u^2 - 1)h = 0.$$

The existence of this positive function implies that u is stable. Indeed, for every smooth compactly supported function φ,

$$\int_{\mathbb{R}^8} \left(|\nabla \varphi|^2 + (3u^2-1)\varphi^2 \right) dx = \int_{\mathbb{R}^8} h^2 \left| \nabla \left(\frac{\varphi}{h} \right) \right|^2 \, dx \geq 0.$$

This identity explains why stability naturally appears in the proof of a theorem about monotonicity.

Since u is bounded and increasing in x_8, the limits

$$v_\pm(x') = \lim_{t \to \pm\infty} u(x',t)$$

exist for every $x' \in \mathbb{R}^7$.

These limiting functions solve the Allen–Cahn equation in seven dimensions and are stable. The new classification therefore tells us that each of them must be either a constant phase or a planar transition.

The proof then shows that these limiting profiles force the original solution to minimize the Allen–Cahn energy locally.

By examining the solution at increasingly large spatial scales, the authors obtain a limiting minimal hypersurface. The monotonicity assumption imposes a special geometric condition on this limit, and a rigidity argument forces it to be a hyperplane.

Finally, a quantitative regularity theorem allows them to return from the large-scale limit to the original solution and conclude that all its level sets are hyperplanes.

The hardest part of the argument is the seven-dimensional stability theorem. Its proof develops new estimates for the geometry of the transition regions, including their curvature, their possible interactions, and their energy growth. A delicate comparison between these estimates rules out any nonplanar stable solution.

This also explains why the two dimensional thresholds differ. Nonplanar stable Allen–Cahn solutions are already known in dimension eight (Pacard and Wei 2013), whereas the new theorem shows that strictly monotone solutions remain planar through that dimension.

The distinction between stability and monotonicity is therefore essential.

The new paper is currently available as a preprint, and its proof awaits independent mathematical scrutiny. If confirmed, it provides a sharp resolution of De Giorgi’s conjecture, nearly fifty years after it was proposed.

The conclusion is especially beautiful: a nonlinear partial differential equation in as many as eight variables, together with the simple requirement that its solution increases in one direction, forces that solution to depend on only one linear coordinate.

References

De Giorgi, Ennio. 1979. “Convergence Problems for Functionals and Operators.” In *Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978)*. Pitagora.

Liu, Yong, Tianci Luo, Kelei Wang, Juncheng Wei, Yong Wei, and Ke Wu. 2026. *Stable De Giorgi Conjecture of the Allen–Cahn Equation in* $\mathbb{R}^3$. https://arxiv.org/abs/2609.21680.

OpenAI. 2026. *A Positive Resolution of De Giorgi's Conjecture in Dimension Eight*. Preprint. https://github.com/openai/math/blob/main/preprints/De-Giorgis-conjecture-in-dimension-eight-September-26-2026/article.pdf.

Pacard, Frank, and Juncheng Wei. 2013. "Stable Solutions of the Allen–Cahn Equation in Dimension 8 and Minimal Cones." *Journal of Functional Analysis* 264 (5): 1131–67. https://doi.org/10.1016/j.jfa.2012.03.010.

Pino, Manuel del, Michał Kowalczyk, and Juncheng Wei. 2011. "On De Giorgi's Conjecture in Dimension $N \geq 9$." *Annals of Mathematics*, 2nd series, vol. 174 (3): 1485–569. https://doi.org/10.4007/annals.2011.174.3.3.

Savin, Ovidiu. 2009. "Regularity of Flat Level Sets in Phase Transitions." *Annals of Mathematics*, 2nd series, vol. 169 (1): 41–78. https://doi.org/10.4007/annals.2009.169.41.