Hilbert’s Sixteenth Problem: A Bound on Limit Cycles

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In September 2026, OpenAI released a preprint (OpenAI 2026) announcing a proof of the uniform boundedness part of Hilbert’s sixteenth problem, one of the famous 23 problems proposed by David Hilbert in 1900. The problem asks a remarkably simple question: *how many limit cycles can a system of two polynomial differential equations have?*

Let us recall the definitions.

Consider a system of differential equations

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases}$$

where P and Q are real polynomials, and the dot denotes differentiation with respect to time.

For example, consider

$$\dot{x} = -y, \qquad \dot{y} = x.$$

The solutions of this system are

$$x(t) = a\cos t - b\sin t, \qquad y(t) = a\sin t + b\cos t,$$

where (a,b) is the initial position.

Thus, every point other than the origin moves around a circle, completing one revolution in time 2π.

More generally, the equations above define a *vector field* in the plane: at each point (x,y), the vector $(P(x,y), Q(x,y))$ indicates the direction and speed of motion.

A solution $(x(t), y(t))$ is called *periodic* if there exists $T > 0$ such that

$$x(t+T) = x(t), \qquad y(t+T) = y(t)$$

for every t. If the solution is not constant, its trajectory is called a *periodic orbit*.

In the example above, every circle centered at the origin is a periodic orbit.

However, the periodic orbits of this system are not isolated: arbitrarily close to any one of them there are infinitely many others.

This motivates the central definition.

A *limit cycle* is a periodic orbit that is *isolated* among periodic orbits. In other words, a periodic orbit Γ is a limit cycle if there exists an open neighbourhood U containing Γ such that no other periodic orbit is entirely contained in U.

The distinction is important. Our first example has infinitely many periodic orbits but no limit cycles.

Now consider a slightly more complicated system:

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2), \\ \dot{y} = x + y(1 - x^2 - y^2). \end{cases}$$

Introducing polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

we obtain

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = 1.$$

The second equation says that every nonstationary trajectory rotates around the origin with constant angular speed.

The first equation describes what happens to its distance r from the origin.

If $0 < r < 1$, then $\dot{r} > 0$, so the trajectory moves outward. If $r > 1$, then $\dot{r} < 0$, so the trajectory moves inward.

On the circle $r = 1$, we have $\dot{r} = 0$, so the trajectory remains on that circle forever.

Consequently, the unit circle

$$x^2 + y^2 = 1$$

is a periodic orbit. Moreover, all nearby trajectories spiral towards it.

This is an example of a *stable limit cycle*: an isolated periodic orbit attracting nearby trajectories.

The system consists of polynomials of degree 3 and has exactly one limit cycle.

Other systems may have several limit cycles, some attracting nearby trajectories and others repelling them.

The central question is how many such cycles can occur.

In the second part of his sixteenth problem, Hilbert (Hilbert 1902) asked for the maximum possible number of limit cycles of a planar polynomial system of a given degree, together with a description of their possible arrangements.

Let

$$H(d) = \sup N(P, Q),$$

where $N(P, Q)$ denotes the number of limit cycles of the system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y),$$

and the supremum is taken over all real polynomials P, Q of degree at most d.

Thus, $H(d)$ measures the largest possible number of limit cycles among *all* polynomial systems of degree at most d.

For linear systems, we have

$$H(1) = 0.$$

Indeed, a linear system may have infinitely many periodic orbits, as in our first example, but none of them is isolated.

For quadratic systems, the problem becomes much harder. Examples with four limit cycles have been constructed, so

$$H(2) \geq 4.$$

However, the exact value of $H(2)$ is not known.

For a long time, it was not even known whether $H(2)$ was finite.

There are two fundamentally different questions here.

The first asks whether *each individual polynomial system* has only finitely many limit cycles.

The second asks whether there is a *uniform bound* on the number of limit cycles for all polynomial systems of a fixed degree.

The second statement is much stronger. Even if every individual system has finitely many limit cycles, the number could, in principle, become arbitrarily large as the coefficients of the polynomials vary.

For example, a family of systems of degree 2 might conceivably have 10, 100, 1000, or even more limit cycles as we change its coefficients, without any individual system having infinitely many.

In 1923, Dulac claimed to prove that every polynomial vector field has finitely many limit cycles. However, a gap was subsequently discovered in his argument.

The individual finiteness problem was finally resolved around 1990 by Ilyashenko (Il'yashenko 1991) and Écalle (Écalle 1992), who developed powerful techniques involving complex analysis and asymptotic expansions.

Their results show that

$$N(P,Q) < \infty$$

for every fixed pair of polynomials P, Q.

But they do not provide a bound depending only on the degrees of P and Q.

Thus, the uniform boundedness part of Hilbert's sixteenth problem remained open for more than a century.

The new OpenAI preprint (OpenAI 2026) claims to resolve this question.

Theorem 1 (OpenAI) *For every integer $d \geq 1$, there exists a finite number $B(d)$ such that every real planar polynomial system*

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases}$$

with $\deg P, \deg Q \leq d$, has at most $B(d)$ limit cycles in the entire plane.
The important point is that the bound $B(d)$ depends *only on the degree* d, not on the coefficients of the polynomials.

There are no assumptions about the sizes or signs of the coefficients, the locations of the cycles, or their stability.

For example, the theorem asserts that there is some finite number $B(2)$ such that *every* quadratic planar polynomial system has at most $B(2)$ limit cycles.

It is worth emphasizing what the theorem does not provide.

The proof establishes the *existence* of $B(d)$, but it does not give an effective formula for it. In particular, it does not determine the exact value of $H(2)$, or of $H(d)$ for any $d \geq 2$.

Nor does it classify the possible arrangements of limit cycles, which was another part of Hilbert's original question.

Nevertheless, establishing uniform finiteness is a major step.

Let us briefly discuss the proof.

One of the main tools in the theory of limit cycles is the *Poincaré return map*.

Suppose we have a periodic orbit Γ and choose a small line segment crossing it transversely. Starting from a point r on this segment, follow the corresponding trajectory until it returns to the segment.

This defines a function

$$\Pi(r),$$

called the Poincaré return map.

A nearby trajectory is periodic precisely when its starting point is a fixed point of this map:

$$\Pi(r) = r.$$

Thus, limit cycles correspond to isolated zeros of the function

$$D(r) = \Pi(r) - r,$$

called the *displacement function*.

This transforms a problem about closed trajectories of differential equations into a problem about zeros of functions.

However, the difficulty is that the return map depends on the coefficients of the polynomial vector field. As these coefficients vary, the trajectories and their return maps can behave in very complicated ways.

In particular, periodic orbits may approach singular points of the vector field, where $P(x, y) = Q(x, y) = 0$. Near such points, trajectories may spend extremely long periods of time before moving away.

The return maps can then develop subtle asymptotic behaviour involving logarithms, exponentials, and other functions that are much more complicated than polynomials.

The authors develop a method for controlling these asymptotic expansions uniformly, including situations where several different scales become large at the same time.

A central idea is that, even when the relevant expressions become extremely complicated, their possible patterns can be reduced to finitely many geometric configurations when the degree of the original vector field is fixed.

The proof then uses analytic estimates and a finite-dimensional counting argument to show that the number of isolated zeros arising from these configurations is uniformly bounded.

A further perturbation argument handles limit cycles that are not stable under small changes of the equations, while a rescaling argument ensures that the bound applies to cycles anywhere in the plane, not just those contained in a fixed bounded region.

Combining these ingredients gives a bound depending only on d.

The resulting proof is highly technical, but its central strategy is appealing: reduce the global problem about periodic trajectories to the local problem of counting zeros of return maps, and then show that polynomial vector fields of bounded degree cannot produce arbitrarily many isolated zeros, even under complicated degenerations.

The preprint is currently awaiting independent mathematical verification. If the proof is confirmed, it will settle the uniform boundedness assertion in the second part of Hilbert's sixteenth problem, posed more than 125 years ago.

The final conclusion is remarkable. Although the behaviour of polynomial differential equations can be extremely complicated, the degree alone imposes a universal bound on the number of isolated periodic motions that can occur.

References

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