The Quasi-Riemann Hypothesis Is Proved

Bogdan Grechuk • 12 Nov 2026

https://functor.network/user/3333/entry/2069

In September 2026, OpenAI released a preprint (OpenAI 2026) announcing a proof of the *quasi-Riemann hypothesis*, a famous weakening of the Riemann Hypothesis. The main result establishes that the Riemann zeta function has no zeros in the half-plane

$$\mathrm{Re}(s) > \frac{7}{8}.$$

Although this statement is weaker than the Riemann Hypothesis, it represents a major advance in our understanding of the distribution of prime numbers.

Let us recall the problem.

For a real number $s > 1$, the *Riemann zeta function* is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots .$$

For example, Euler famously proved that

$$\zeta(2) = \frac{\pi^2}{6}.$$

The zeta function can also be defined for complex numbers $s = \sigma + it$, where $\sigma, t \in \mathbb{R}$ and $i^2 = -1$, by the same series whenever $\sigma > 1$.

One of Euler's most important discoveries was the formula

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}}, \qquad \mathrm{Re}(s) > 1,$$

where the product runs over all prime numbers.

This formula, called the *Euler product*, reveals the close connection between the zeta function and prime numbers. In particular, it implies that

$$\zeta(s) \neq 0 \qquad \text{whenever } \mathrm{Re}(s) > 1.$$

The function can be extended, by a process called *analytic continuation*, to all complex numbers except $s = 1$, where it has a simple pole.

The extended function has zeros at

$$s = -2, -4, -6, \ldots,$$

called the *trivial zeros*. All other zeros lie in the strip

$$0 < \mathrm{Re}(s) < 1,$$

called the *critical strip*.

In 1859, Riemann conjectured that all these nontrivial zeros lie exactly on the vertical line

$$\mathrm{Re}(s) = \frac{1}{2}.$$

This is the famous *Riemann Hypothesis*, one of the most important unsolved problems in mathematics.

But why are these zeros so important?

The answer is that their positions determine how accurately we can count prime numbers.

Let $\pi(x)$ denote the number of primes not exceeding x. For example,

$$\pi(10) = 4,$$

because there are four primes $2, 3, 5, 7$ not exceeding 10.

The *prime number theorem*, proved independently by Hadamard (Hadamard 1896) and de la Vallée Poussin (Vallée Poussin 1896) in 1896, states that

$$\pi(x) \sim \frac{x}{\log x},$$

where $\log$ denotes the natural logarithm and the symbol $\sim$ means that the ratio of the two expressions tends to 1 as $x \to \infty$.

A more accurate approximation uses the *logarithmic integral*

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

The prime number theorem is equivalent to

$$\pi(x) \sim \mathrm{Li}(x).$$

However, this tells us only that the relative error tends to zero. It does not tell us how quickly.

The answer depends on the zeros of the Riemann zeta function.

Riemann discovered a remarkable formula connecting prime numbers with these zeros. Roughly speaking, the error in counting primes can be expressed as a sum of contributions from the nontrivial zeros

$$\rho = \beta + i\gamma$$

of $\zeta(s)$.

A zero with real part β contributes an oscillating term whose size is related to x^{β}.

Thus, the farther the zeros are from the line $\mathrm{Re}(s) = 1$, the smaller the errors we can expect in the prime number theorem.

In particular, the Riemann Hypothesis would imply the famous estimate

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\log x),$$

where $O(g(x))$ means that the absolute value of the error is bounded by a constant multiple of $g(x)$ for all sufficiently large x.

This would be an extraordinarily accurate approximation.

However, the Riemann Hypothesis remains unproved.

In 1896, Hadamard and de la Vallée Poussin established the weaker statement that $\zeta(s)$ has no zeros on the line

$$\mathrm{Re}(s) = 1.$$

This was enough to prove the prime number theorem.

In 1899, de la Vallée Poussin (Vallée Poussin 1899) improved this by proving that there exists a constant $c > 0$ such that $\zeta(s)$ has no zeros in the region

$$\mathrm{Re}(s) > 1 - \frac{c}{\log(|\mathrm{Im}(s)| + 3)}.$$

Here $\mathrm{Im}(s)$ denotes the imaginary part of s.

This gives a region extending a little to the left of the line $\mathrm{Re}(s) = 1$. However, its width decreases as the imaginary part of s increases.

Later, Vinogradov and Korobov obtained significantly larger zero-free regions. Nevertheless, their width still tends to zero at large heights.

Consequently, these classical results do not exclude the possibility that some zeros have real parts arbitrarily close to 1.

This leads to the following question: *is there a fixed constant $\sigma_0 < 1$ such that every nontrivial zero ρ of the Riemann zeta function satisfies*

$$\mathrm{Re}(\rho) \leq \sigma_0?$$

This statement is known as the *quasi-Riemann hypothesis*.

It is much weaker than the Riemann Hypothesis, because it does not require the zeros to lie on one particular line. It merely requires that they remain a fixed positive distance away from $\mathrm{Re}(s) = 1$.

However, this apparently modest weakening resisted proof for more than a century.

The new OpenAI preprint (OpenAI 2026) establishes the following result.

Theorem 1 (OpenAI) *The Riemann zeta function has no zeros in the half-plane*

$$\mathrm{Re}(s) > \frac{7}{8}.$$

More precisely,

$$\zeta(s) \neq 0 \qquad \textit{for all } s \neq 1 \textit{ with } \mathrm{Re}(s) > \frac{7}{8}.$$

Thus, every nontrivial zero ρ satisfies

$$\mathrm{Re}(\rho) \leq \frac{7}{8}.$$

Moreover, the functional equation of the zeta function implies that the nontrivial zeros are symmetric with respect to the line $\mathrm{Re}(s) = 1/2$.

Consequently, the theorem places all nontrivial zeros in the narrower strip

$$\frac{1}{8} \leq \mathrm{Re}(\rho) \leq \frac{7}{8}.$$

The Riemann Hypothesis predicts the much stronger conclusion that

$$\mathrm{Re}(\rho) = \frac{1}{2}.$$

Therefore, the new theorem does not settle the Riemann Hypothesis itself.

Nevertheless, it completely resolves the quasi-Riemann hypothesis by establishing a fixed zero-free half-plane.

The preprint actually proves a stronger result: the same zero-free half-plane applies to *every Dirichlet L-function*, a generalization of the Riemann zeta function. We will concentrate here on the consequences for $\zeta(s)$.

The most important application is an improved error bound in the prime number theorem.

Using classical formulas relating the zeros of $\zeta(s)$ to the distribution of primes, the theorem implies

$$\boxed{\pi(x) = \mathrm{Li}(x) + O(x^{7/8}\log x).}$$

This is a *power-saving error term*: the exponent of x is strictly smaller than 1.

To understand the improvement, recall that

$$\mathrm{Li}(x) \sim \frac{x}{\log x}.$$

Thus, the relative error is at most of order

$$\frac{x^{7/8}\log x}{x/\log x} = x^{-1/8}(\log x)^2.$$

This tends to zero at a polynomial rate, apart from the logarithmic factor.

By comparison, the classical zero-free regions give error terms whose savings are smaller than every fixed positive power of x.

Thus, the new theorem substantially improves our knowledge of how accurately the logarithmic integral approximates the number of primes.

Let us briefly discuss the proof.

The authors do not study the Riemann zeta function in isolation. Instead, they investigate a larger family of related functions, using ideas from analytic number theory.

An important ingredient is the theory of *theta functions*, which are special functions defined by infinite sums and possessing remarkable transformation properties.

The proof constructs carefully chosen sums that can be analyzed in two different ways. One uses transformation formulas for theta functions and a technique called *Poisson summation*. The other expresses these sums through integrals involving reciprocals of the relevant L-functions.

If an L-function had a zero too close to the line $\mathrm{Re}(s) = 1$, its reciprocal would develop a singularity. The authors derive estimates showing that such a singularity would be incompatible with the two descriptions of their sums.

The argument proceeds in two stages. The first establishes a weaker zero-free half-plane

$$\mathrm{Re}(s) > \frac{11}{12}.$$

The second introduces additional estimates and improves the boundary to $7/8$.

The proof is technically demanding, involving estimates for character sums and moments of L-functions. The main conceptual idea is to combine several different representations of the same arithmetic expressions to rule out zeros in a fixed region.

The numerical constant $7/8$ is not expected to be optimal. Indeed, subsequent work by Baiying Liu (Liu 2026) has already improved it slightly. However, the essential achievement is not the particular fraction but the existence of *any* fixed zero-free half-plane.

The quasi-Riemann hypothesis is an appealing example of how a weakened version of a famous conjecture can remain extremely difficult while still having powerful consequences.

The Riemann Hypothesis would give an almost square-root error in counting primes. The new theorem does not go that far, but it establishes for the first time a fixed power saving in the prime number theorem through a universal bound on the real parts of zeta zeros.

The preprint is subject to the usual process of mathematical scrutiny. If its proof is confirmed, it will mark a major advance in one of the central problems of analytic number theory.

References

Hadamard, Jacques. 1896. “Sur La Distribution Des zéros de La Fonction $\zeta(s)$ Et Ses Conséquences Arithmétiques.” *Bulletin de La Société Mathématique de France* 24: 199–220. https://doi.org/10.24033/bsmf.545.

Liu, Baiying. 2026. *Slightly Improved Zero-Free Half-Planes for the Quasi-Riemann Hypothesis*. https://arxiv.org/abs/2610.12234.

OpenAI. 2026. *The Quasi-Riemann Hypothesis: A Zero-Free Half-Plane* $\mathrm{Re}(s) > 7/8$. Preprint. https://github.com/openai/math/blob/main/preprints/The-Quasi-Riemann-Hypothesis-September-30-2026/paper.pdf.

Vallée Poussin, Charles-Jean de la. 1896. “Recherches Analytiques Sur La Théorie Des Nombres Premiers. Première Partie: La Fonction $\zeta(s)$ de Riemann Et Les Nombres Premiers En général.” *Annales de La Société Scientifique de Bruxelles* 20: 183–256.

Vallée Poussin, Charles-Jean de la. 1899. “Sur La Fonction $\zeta(s)$ de Riemann Et Le Nombre Des Nombres Premiers Inférieurs à Une Limite Donnée.” *Mémoires Couronnés Et Autres mémoires Publiés Par l’Académie Royale de Belgique* 59: 1–74. https://doi.org/10.3406/marb.1899.2449.