Ultraflat Littlewood Polynomials

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In October 2026, OpenAI released a preprint (OpenAI 2026) announcing a proof of the existence of *ultraflat Littlewood polynomials*. The result strengthens a celebrated theorem of Balister, Bollobás, Morris, Sahasrabudhe, and Tiba (Balister et al. 2020), published in the *Annals of Mathematics* in 2020, and resolves a longstanding question in harmonic analysis.

The problem is remarkably easy to formulate: *can a polynomial with coefficients restricted to* -1 *and* 1 *have almost constant absolute value on the unit circle?*

Let us recall the definitions.

A *Littlewood polynomial* is a polynomial of the form

$$P(z) = \sum_{k=0}^{N-1} \epsilon_k z^k, \qquad \epsilon_k \in \{-1, 1\}.$$

Thus, P has degree $N-1$, and each of its N coefficients is either 1 or -1.

We are interested in the values of $P(z)$ when z lies on the *unit circle*

$$|z| = 1.$$

Writing $z = e^{it}$, where $t \in [0, 2\pi)$, we obtain

$$P(e^{it}) = \sum_{k=0}^{N-1} \epsilon_k e^{ikt}.$$

This is a trigonometric sum with very restricted coefficients. Depending on t, its terms may reinforce or cancel one another, so its absolute value can fluctuate considerably.

For example, take

$$P(z) = 1 + z.$$

Then

$$|P(e^{it})| = 2|\cos(t/2)|.$$

Its absolute value varies from 0 to 2. In particular, it is far from constant.

But perhaps, by carefully choosing many signs ϵ_k, we can arrange for the fluctuations to become small?

Before answering this question, let us establish an elementary identity.

Expanding the square and integrating over the unit circle gives

$$\frac{1}{2\pi}\int_0^{2\pi} |P(e^{it})|^2\,dt = \sum_{k=0}^{N-1} |\epsilon_k|^2 = N.$$

Indeed, the integral of $e^{i(k-l)t}$ is zero whenever $k \neq l$, and equals 2π when $k = l$.

This is a special case of *Parseval's identity*.

Consequently,

$$\max_{|z|=1} |P(z)| \geq \sqrt{N}.$$

Thus, $\sqrt{N}$ is the natural scale for the absolute value of a Littlewood polynomial.

Ideally, we would like to find polynomials satisfying

$$|P(z)| \approx \sqrt{N} \qquad \text{for every } |z| = 1.$$

Exact equality is impossible when $N > 1$. Indeed, in the expansion of $|P(e^{it})|^2$, the coefficient of $e^{i(N-1)t}$ is

$$\epsilon_{N-1}\epsilon_0 = \pm 1,$$

whereas all nonconstant Fourier coefficients of a constant function must vanish.

However, it might still be possible to make the relative fluctuations arbitrarily small as N grows.

This question has a long history.

In 1957, Erdős (Erdős 1957) asked whether there exist constants $\delta, \Delta > 0$, independent of N, such that for every sufficiently large N one can choose a Littlewood polynomial satisfying

$$\delta\sqrt{N} \leq |P(z)| \leq \Delta\sqrt{N} \qquad (|z| = 1).$$

Such polynomials are called *flat Littlewood polynomials*.

Notice that the problem asks for *both* inequalities. An upper bound alone is much easier to obtain.

In 1959, Rudin (Rudin 1959), building on earlier work of Shapiro, constructed Littlewood polynomials whose maximum absolute value on the unit circle is at most $\sqrt{2N}$ for lengths that are powers of two. However, these constructions do not provide the required positive lower bound everywhere on the circle.

In 1966, Littlewood (Littlewood 1966) further emphasized the importance of this problem.

An important advance was made by Kahane (Kahane 1980) in 1980. He proved that if the coefficients ϵ_k are allowed to be arbitrary complex numbers of absolute value 1, rather than just -1 or 1, then one can achieve

$$|P(z)| = (1 + o(1))\sqrt{N}$$

uniformly on the unit circle.

Here $o(1)$ denotes a quantity tending to zero as $N \to \infty$.

Such polynomials are called *ultraflat*.

However, Kahane's construction used complex coefficients. The restriction to real signs $\{-1, 1\}$ turned out to be much more difficult.

In 2020, Balister, Bollobás, Morris, Sahasrabudhe, and Tiba (Balister et al. 2020) finally proved the existence of flat Littlewood polynomials, resolving the original Erdős–Littlewood problem.

Their theorem guarantees constants $\delta, \Delta > 0$ for which

$$\delta\sqrt{N} \le |P(z)| \le \Delta\sqrt{N}$$

on the entire unit circle, for every sufficiently large N.

But the constants δ and Δ are fixed. The theorem does not say that they can be made arbitrarily close to 1.

Could the much stronger conclusion of Kahane's theorem also hold when every coefficient is restricted to ± 1?

The new preprint (OpenAI 2026) answers this question affirmatively.

Theorem 1 (OpenAI) *For every $\varepsilon > 0$, there exists an integer N_0 such that for every integer $N \ge N_0$, one can choose signs*

$$\epsilon_0, \dots, \epsilon_{N-1} \in \{-1, 1\}$$

so that the polynomial

$$P(z) = \sum_{k=0}^{N-1} \epsilon_k z^k$$

satisfies

$$(1 - \varepsilon)\sqrt{N} \le |P(z)| \le (1 + \varepsilon)\sqrt{N}$$

for every complex number z with $|z| = 1$.

Thus, Littlewood polynomials can be made *arbitrarily close to having constant absolute value* on the entire unit circle, in relative terms.

The conclusion is optimal in an asymptotic sense. By Parseval's identity, $\sqrt{N}$ is exactly the square root of the mean squared absolute value, so neither side of the inequality could approach a different constant multiple of $\sqrt{N}$.

One important detail is that the signs may be chosen separately for each N. The theorem does not assert that we can choose one infinite sequence of signs such that all its sufficiently long initial segments are ultraflat.

Let us briefly discuss the proof.

The authors do not attempt to choose the signs directly. Instead, they first construct a continuous complex-valued function B_N on the unit circle whose absolute value is very close to 1.

The function is constructed so that its Fourier coefficients are almost entirely concentrated in the consecutive frequencies

$$0, 1, \ldots, N-1.$$

Moreover, these Fourier coefficients are real and have absolute values at most approximately $1/\sqrt{N}$.

This is achieved by combining carefully designed oscillatory waves. Their phases are adjusted so that the absolute value of the resulting function remains nearly constant, while its Fourier coefficients satisfy the required bounds.

At this stage, the coefficients are real numbers rather than signs. How can they be replaced by -1 or 1 without destroying the almost constant absolute value?

Here Parseval's identity provides an important clue.

Since $|B_N|$ is close to 1, the sum of the squares of its Fourier coefficients is close to 1. On the other hand, the coefficients at the N relevant frequencies have absolute values bounded by approximately $1/\sqrt{N}$.

Consequently, most of these coefficients must already be close in absolute value to $1/\sqrt{N}$. After multiplying them by $\sqrt{N}$, most are close to -1 or 1.

The remaining task is to round all the coefficients to signs while keeping the resulting change in the polynomial small *simultaneously at every point of the unit circle*.

Independent random rounding would not provide the required uniform control. Instead, the authors use techniques from *discrepancy theory*, which studies how to assign signs to many numbers so that large collections of sums remain small.

A discrepancy estimate allows the coefficients to be rounded to ± 1 with an error that becomes negligible compared with $\sqrt{N}$. This completes the construction.

The theorem also has an interesting interpretation in terms of binary sequences.

Given the signs $\epsilon_0, \ldots, \epsilon_{N-1}$, define their *aperiodic autocorrelations*

$$C_j = \sum_{k=0}^{N-1-j} \epsilon_k \epsilon_{k+j}, \qquad 1 \le j < N.$$

These quantities measure how strongly the sequence correlates with shifted copies of itself.

Expanding $|P(e^{it})|^2$ gives

$$|P(e^{it})|^2 = N + 2 \sum_{j=1}^{N-1} C_j \cos(jt).$$

By orthogonality of trigonometric functions,

$$\frac{1}{2\pi} \int_0^{2\pi} \left(|P(e^{it})|^2 - N \right)^2 dt = 2 \sum_{j=1}^{N-1} C_j^2.$$

For an ultraflat polynomial, the left-hand side is $o(N^2)$. Therefore,

$$\sum_{j=1}^{N-1} C_j^2 = o(N^2).$$

Thus, the theorem also produces long sequences of signs with exceptionally small total squared autocorrelation.

This has consequences for the study of the *merit factor* of binary sequences, defined by

$$F_N = \frac{N^2}{2 \sum_{j=1}^{N-1} C_j^2}.$$

The theorem implies that there exist sequences for which F_N tends to infinity as N grows.

The result therefore connects a classical question about polynomials with problems concerning binary sequences and their correlation properties.

The preprint is awaiting independent mathematical verification. If its proof is confirmed, it will provide a remarkably sharp resolution of the flatness problem for Littlewood polynomials.

The conclusion is especially striking because the coefficients have almost no freedom: each must be exactly -1 or 1. Nevertheless, with sufficiently many carefully chosen signs, their complex oscillations can combine to produce a polynomial whose absolute value is almost constant at every point of the unit circle.

References

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