Koebe’s Circle-Domain Conjecture Is Proved

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In September 2026, a preprint (OpenAI 2026) announced a proof of Koebe’s circle-domain conjecture, a classical problem in complex analysis posed by Paul Koebe in 1908. The conjecture asks whether every domain in the complex plane can be transformed, by a conformal map, into a domain whose holes are all round disks or single points.

Let us recall the problem.

A *domain* is a nonempty connected open set. We will work in the *Riemann sphere*

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\},$$

obtained by adding one point at infinity to the complex plane.

A function f is called *holomorphic* if it is complex differentiable at every point of its domain. A *conformal equivalence* between two domains D and Ω is a bijective holomorphic map

$$f : D \longrightarrow \Omega.$$

On the Riemann sphere, holomorphicity is understood in the usual local coordinates.

Conformal maps preserve angles between smooth curves. However, they need not preserve distances, areas, or the shapes of regions.

The famous *Riemann mapping theorem* states that every simply connected domain in $\mathbb{C}$, other than the whole plane itself, is conformally equivalent to the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Thus, even if the boundary of a simply connected domain has an extremely complicated shape, a conformal map can transform it into a domain with a perfectly circular boundary.

But what happens if the domain has several holes?

For example, consider the annulus

$$A = \{z \in \mathbb{C} : 1 < |z| < 2\}.$$

Its complement in the Riemann sphere has two connected components:

$$\{z : |z| \leq 1\} \quad \text{and} \quad \{z : |z| \geq 2\} \cup \{\infty\}.$$

Both are closed round disks on the sphere.

More generally, a *circle domain* is a domain $\Omega \subseteq \widehat{\mathbb{C}}$ such that every connected component of its complement

$$\widehat{\mathbb{C}} \setminus \Omega$$

is either a closed round disk or a single point.

Single points are included because a domain may have punctures, which can be regarded as disks of radius zero.

Notice that a circle domain may have infinitely many complementary components. In particular, these components need not be separated by any fixed positive distance.

In 1908, Koebe (Koebe 1908) conjectured that every domain is conformally equivalent to a circle domain.

In other words, no matter how complicated the geometry of the holes is, it should be possible to make all of them round simultaneously.

For domains with only finitely many complementary components, Koebe himself proved the conjecture; see his 1920 paper (Koebe 1920).

However, the situation becomes much more difficult when the number of holes is infinite.

In 1993, He and Schramm (He and Schramm 1993) made a major breakthrough by proving Koebe’s conjecture for all domains with at most countably many complementary components.

This was a substantial extension of Koebe’s original result. Nevertheless, arbitrary domains can have *uncountably many* complementary components.

For example, take a Cantor set $C \subseteq [0,1]$ and remove from the plane the set

$$C \times [0,1].$$

The complement is a connected open set, but the removed set consists of uncountably many disjoint vertical segments, each forming a separate connected component.

Such examples illustrate why countable connectivity is a genuine restriction.

Further progress was obtained by imposing geometric conditions on the holes or on the domain itself. In particular, Esmayli and Rajala (Esmayli and Rajala 2026) proved uniformization results under quasitripod conditions, while Karafyllia and Ntalampekos (Karafyllia and Ntalampekos 2026) established the conjecture for a class of Gromov-hyperbolic domains.

The new preprint (OpenAI 2026) removes all these additional assumptions.

Theorem 1 (OpenAI) *Every domain $D \subseteq \widehat{\mathbb{C}}$ is conformally equivalent to a circle domain.*
More precisely, there exists a domain $\Omega \subseteq \widehat{\mathbb{C}}$ and a conformal bijection

$$f : D \longrightarrow \Omega$$

such that every connected component of

$$\widehat{\mathbb{C}} \setminus \Omega$$

is either a closed round disk or a single point.
Thus, every domain on the Riemann sphere, however complicated its boundary or the arrangement of its complementary components, has a conformal model with only circular holes and punctures.

There is an interesting consequence of the theorem when the domain has uncountably many complementary components.

Any collection of pairwise disjoint nondegenerate open disks in the plane is at most countable, because each disk contains a point with rational coordinates.

Consequently, in a circle domain, at most countably many complementary components can be nondegenerate closed disks. All the remaining components must be single points.

Therefore, the theorem implies that even if the original domain has uncountably many complicated complementary components, there is a conformal model in which all but countably many of them are reduced to points.

This is one of the most striking features of the result.

Let us briefly discuss the proof.

The natural starting point is to approximate a general domain by simpler domains with finitely many holes. Each of these can be transformed into a circle domain by Koebe’s classical theorem.

One might hope that, by taking a limit of these conformal maps, one would obtain the required conformal equivalence for the original domain.

Unfortunately, this argument has a serious difficulty.

Even if every approximating complementary component is a round disk, the limiting component need not be exactly the limiting disk. Extra parts of the complement may remain attached to it, producing a component that is no longer round.

Furthermore, some complementary components may be so small or irregular that they are not adequately controlled by following individual disks through the approximations.

The main challenge is therefore to control *all* complementary components simultaneously, including uncountably many of them.

The authors develop a method based on special analytic functions, called *test functions*, associated with the boundary components. Their behaviour is measured using the *Dirichlet energy*

$$\int_D |\nabla u|^2 \, dA,$$

where u is a real-valued function and ∇u denotes its gradient.

The proof uses two different types of tests.

The first type consists of functions that act as barriers. These functions allow the authors to show that almost all complementary components must collapse to single points in the limiting domain.

More precisely, they identify a countable collection of exceptional components and construct tests forcing every other component to become a point.

The second type of test deals with these exceptional components. It uses functions whose values change by a prescribed amount when one travels around a selected hole.

Such functions can detect whether the limiting component extends beyond its intended round disk. An integral estimate then rules out this possibility.

A key ingredient is a theory of weighted paths and boundary intersections, related to the *transboundary extremal length* introduced by Schramm (Schramm 1995). The authors use this theory to obtain estimates that remain valid even when the boundary has uncountably many components.

Combining the two types of tests ensures that every complementary component of the limiting domain is either exactly a round disk or exactly one point.

This completes the proposed proof of Koebe’s conjecture.

There is an important subtlety in the conclusion. The conformal map is defined on the original domain itself. The theorem does not assert that it extends to a homeomorphism of the whole Riemann sphere transforming the original holes into round disks.

In particular, the shapes of the original holes may be extremely complicated; the theorem provides a conformally equivalent domain with simpler holes, rather than a global deformation of the entire plane.

This distinction is relevant to our earlier blog post on quasiconformal uniformization of Sierpiński carpets. Quasiconformal maps allow a controlled amount of angle distortion, whereas conformal maps preserve angles exactly. On the other hand, the earlier result concerns global homeomorphisms of the sphere, while Koebe's theorem concerns conformal equivalences defined on domains.

The new theorem also settles an existence question, not a general uniqueness question. Whether two circle domains that are conformally equivalent must be related by a Möbius transformation requires additional hypotheses.

Koebe's conjecture is a particularly beautiful example of a uniformization problem: it asks whether an arbitrary geometric object can be represented in a remarkably simple standard form without changing its intrinsic complex-analytic structure.

The conjecture remained open for 118 years. The new preprint claims that, at least as far as conformal equivalence is concerned, every planar domain can be represented by one whose complementary components are nothing more complicated than round disks and points.

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