Weak Mixing in Every Irrational Triangle

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In May 2026, I wrote a blog post about a theorem of Chaika and Forni (Chaika and Forni 2026), published in the *Annals of Mathematics*, proving that there exist polygons whose billiard flows are weakly mixing. In fact, they showed that such polygons form a dense set among polygons with any fixed number of vertices.

In October 2026, OpenAI released a preprint (OpenAI 2026) announcing a considerably stronger result for triangles: every triangle with at least one irrational angle has a weakly mixing billiard flow.

Let us recall the problem.

Consider a billiard ball moving inside a polygon Q. The ball travels in straight lines at constant speed and reflects from the sides according to the usual rule: the angle of incidence equals the angle of reflection.

We ignore trajectories that hit a vertex of the polygon, where the reflection rule is not well-defined. Such trajectories form a set of measure zero.

To describe the motion mathematically, we need to specify both the position $x \in Q$ of the ball and its direction, represented by an angle $\theta \in [0, 2\pi)$. Thus, the *phase space* of the billiard consists of pairs (x, θ).

Let $\Phi^t(x, \theta)$ denote the position and direction of the ball after time t, assuming unit speed. The family of transformations $\{\Phi^t\}_{t \in \mathbb{R}}$ is called the *billiard flow*.

The natural probability measure on this phase space is

$$d\mu(x, \theta) = \frac{dA(x)\, d\theta}{2\pi \operatorname{Area}(Q)},$$

where dA is the usual area measure. This means that the initial position is chosen uniformly inside the polygon and the initial direction uniformly on the circle.

The billiard flow preserves this measure.

A measure-preserving flow is called *ergodic* if every measurable subset invariant under the flow has measure 0 or 1. Informally, an ergodic system cannot be decomposed into two smaller invariant regions of positive measure.

A stronger property is *weak mixing*. One way to define it is to consider two independent copies of the system, evolving simultaneously. The original flow is weakly mixing if this product flow is ergodic.

Equivalently, for every pair of measurable subsets A, B of the phase space,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left|\mu(A \cap \Phi^{-t}B) - \mu(A)\mu(B)\right|^2 \, dt = 0.$$

Thus, observations made at widely separated times become asymptotically independent in an averaged sense.

The even stronger property of *mixing* requires

$$\mu(A \cap \Phi^{-t}B) \longrightarrow \mu(A)\mu(B)$$

as $t \to \infty$, without averaging.

We have the implications

$$\text{mixing} \implies \text{weak mixing} \implies \text{ergodicity}.$$

For polygonal billiards, even the weakest of these properties is difficult to establish.

A polygon is called *rational* if all its interior angles are rational multiples of π. For example, equilateral and right isosceles triangles are rational.

In rational polygons, the full billiard flow cannot be ergodic. Indeed, reflections in the sides generate only finitely many possible changes of direction, so the motion preserves nontrivial information about its direction.

Nevertheless, a famous theorem of Kerckhoff, Masur, and Smillie (Kerckhoff et al. 1986), proved in 1986, states that in every rational polygon, the billiard flow restricted to almost every direction is uniquely ergodic.

For irrational polygons, the full billiard flow can be ergodic. Some examples had been known since the 1980s, and Vorobets (Vorobets 1997) subsequently established ergodicity under certain quantitative approximation conditions.

However, whether the full billiard flow could be weakly mixing remained a long-standing question until the theorem of Chaika and Forni (Chaika and Forni 2026).

Their result showed that weak mixing is possible and, moreover, occurs for a dense family of polygons. But a dense family need not contain every polygon, or even almost every polygon in the sense of Lebesgue measure.

The new preprint (OpenAI 2026) establishes a much stronger conclusion for triangles.

Theorem 1 (OpenAI) *Let Q be a nondegenerate Euclidean triangle with at least one interior angle that is an irrational multiple of π. Then the unit-speed billiard flow in Q is weakly mixing with respect to the normalized measure*

$$d\mu(x,\theta) = \frac{dA(x)\,d\theta}{2\pi\,\mathrm{Area}(Q)}.$$

In other words, the theorem applies to *every irrational triangle*, without any additional condition on its angles.

For example, it applies to every right triangle whose acute angles are irrational multiples of π. Previously, the behaviour of billiards in such triangles was understood only under special assumptions.

The theorem also implies that almost every triangle has a weakly mixing billiard flow. In fact, up to similarity, a triangle is determined by two of its angles, and triangles whose angles are all rational multiples of π form only a countable set.

Thus, in this two-dimensional space of triangle shapes, the theorem has only countably many exceptions.

Moreover, these exceptions are necessary: as explained above, the full billiard flow of a rational triangle is not even ergodic. Therefore, the condition on the angles cannot be weakened.

Let us briefly discuss the proof.

The starting point is the classical technique of *unfolding* billiard trajectories. Instead of reflecting the moving ball from a side, we reflect the polygon itself and let the trajectory continue along a straight line.

The authors glue two oppositely oriented copies of the triangle along their corresponding sides. The result is a flat surface except at three special points, corresponding to the vertices of the triangle. Near these points, the surface looks like a cone, so they are called *conical singularities*.

The main analytic step concerns functions on the space of positions and directions satisfying

$$f(\Phi^t(x,\theta)) = e^{i\lambda t} f(x,\theta)$$

for some real number λ.

Such functions are called *eigenfunctions* of the flow. A standard theorem from ergodic theory says that a measure-preserving flow is weakly mixing if and only if it has no nonconstant eigenfunctions.

The authors prove a strong regularity statement for these eigenfunctions. Using Fourier expansions in the angular variable, together with delicate estimates near the conical singularities, they show that every bounded eigenfunction must have the form

$$f(x,\theta) = c(\theta)e^{i\lambda x\cdot v(\theta)}$$

inside the triangle, where

$$v(\theta) = (\cos\theta, \sin\theta)$$

and c depends only on the direction.

This is the difficult analytic part of the argument: the eigenfunction is initially only measurable, so one cannot simply differentiate it. The authors establish the required regularity using smoothing and energy estimates, building on techniques developed by Forni and Moll (Forni and Moll 2025).

Once this representation has been obtained, the rest of the proof becomes an elegant geometric argument.

Choose a vertex whose interior angle α is an irrational multiple of π, and place that vertex at the origin.

Reflections in the two sides meeting at this vertex imply that $c(\theta)$ is invariant under rotation through the angle 2α.

Because α/π is irrational, repeated rotations through 2α are dense on the circle. Consequently, a measurable function invariant under this rotation must be constant almost everywhere.

Thus, $c(\theta)$ is constant.

Finally, reflection in the third side of the triangle forces $\lambda = 0$. Therefore, every eigenfunction is constant, which proves weak mixing.

The theorem does not establish the still stronger property of mixing. Whether any polygonal billiard flow can be mixing remains an open question.

There is also no contradiction with the existence of periodic billiard trajectories. As discussed in another previous post, Forni proved in 2026 that every polygon has a periodic billiard trajectory. Such trajectories can coexist with weak mixing because they form a set of measure zero.

The new result substantially advances our understanding of one of the simplest dynamical systems. It shows that, for triangular billiards, the distinction between rational and irrational angles is decisive: rational triangles have non-ergodic full billiard flows, while every irrational triangle has a weakly mixing flow.

The preprint is awaiting independent verification. If confirmed, it provides a remarkably complete answer to the weak-mixing question for triangular billiards.

References

Chaika, Jon, and Giovanni Forni. 2026. “Weakly Mixing Polygonal Billiards.” *Annals of Mathematics*, 2nd series, vol. 203 (3): 695–735. https://doi.org/10.4007/annals.2026.203.3.1.

Forni, Giovanni, and Nelson Moll. 2025. *Cohomological Equation for Geodesic Flows on Flat Surfaces*. https://arxiv.org/abs/2510.18128.

Kerckhoff, Steven, Howard Masur, and John Smillie. 1986. “Ergodicity of Billiard Flows and Quadratic Differentials.” *Annals of Mathematics*, 2nd series, vol. 124 (2): 293–311. https://doi.org/10.2307/1971280.

OpenAI. 2026. *Weak Mixing of Triangular Billiards with an Irrational Angle*. Preprint. https://github.com/openai/math/blob/main/preprints/Weak-mixing-of-triangular-billiards-with-an-irrational-angle-October-5-2026/weak-mixing-triangular-billiards.pdf.

Vorobets, Ya. B. 1997. “Ergodicity of Billiards in Polygons.” *Sbornik: Mathematics* 188 (3): 389–434. https://doi.org/10.1070/SM1997v188n03ABEH000211.