The Erdős–Szemerédi Sunflower Conjecture Is Proved

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In October 2026, Matija Bucić, Fankang He, and Hyunwoo Lee (Bucić et al. 2026) posted a preprint announcing a proof of the Erdős–Szemerédi sunflower conjecture, a problem in extremal combinatorics that had remained open for almost fifty years. The conjecture concerns a simple question: how large can a family of subsets of a finite set be if it contains no sunflower?

Let us recall the definitions.

Suppose that X is a finite set. A collection of r distinct subsets

$$A_1, A_2, \dots, A_r \subseteq X$$

is called an r-*sunflower* if the intersection of every two of these sets is the same:

$$A_i \cap A_j = C \qquad \text{for all } i \neq j,$$

where C is some fixed set, called the *core* of the sunflower.

For example, the three sets

$$\{1,2\}, \qquad \{1,3\}, \qquad \{1,4\}$$

form a 3-sunflower with core $\{1\}$.

The name comes from the fact that, after removing the common core, the remaining parts of the sets, called *petals*, are pairwise disjoint.

The core may also be empty. In particular, any collection of r pairwise disjoint sets forms an r-sunflower.

In 1960, Erdős and Rado (Erdős and Rado 1960) proved a fundamental result about sunflowers. They showed that if a family of sets is sufficiently large, while every set in the family has at most w elements, then the family necessarily contains an r-sunflower.

More precisely, they proved that a family containing more than

$$w!(r-1)^w$$

sets, each of size at most w, must contain an r-sunflower.

They conjectured that the factorial dependence on w could be dramatically improved: for every fixed $r \geq 3$, there should exist a constant C_r such that C_r^w sets suffice.

This became known as the *Erdős–Rado sunflower conjecture*.

The problem attracted considerable attention. In 2021, Alweiss, Lovett, Wu, and Zhang (Alweiss et al. 2021) made a major breakthrough, obtaining a bound of roughly $(C_r \log w)^w$. Subsequent refinements improved the dependence on r and the logarithmic factors. However, the conjectured bound C_r^w remains open.

The theorem we discuss today concerns a related, but different, sunflower conjecture.

Instead of restricting the sizes of the individual sets, suppose that we restrict the total number of elements in the ground set X.

Let $|X| = n$. Since X has exactly 2^n subsets, any family

$$\mathcal{F} \subseteq 2^X$$

has at most 2^n members. Here 2^X denotes the collection of all subsets of X, and $|\mathcal{F}|$ denotes the number of sets in the family.

But what if $\mathcal{F}$ contains no r-sunflower?

In 1970, Erdős (Erdős 1970) recorded the conjecture that, for every fixed $r \geq 3$, there exists a constant $c_r < 2$ such that

$$|\mathcal{F}| \leq c_r^n$$

whenever $\mathcal{F}$ contains no r-sunflower.

The conjecture was subsequently studied by Erdős and Szemerédi (Erdős and Szemerédi 1978), and became known as the *Erdős–Szemerédi sunflower conjecture*.

The important point is that the upper bound is *exponentially smaller* than the total number 2^n of subsets.

In 1978, Erdős and Szemerédi proved the weaker estimate

$$|\mathcal{F}| \leq 2^{n - c\sqrt{n}},$$

where $c > 0$ depends on r.

This shows that the proportion of subsets belonging to a sunflower-free family tends to zero, but it does not give the conjectured exponential decay.

A major breakthrough occurred in 2016, when Croot, Lev, and Pach (Croot et al. 2017), and Ellenberg and Gijswijt (Ellenberg and Gijswijt 2017), developed powerful polynomial methods for studying sets without arithmetic progressions in finite groups.

In particular, Ellenberg and Gijswijt proved the famous *cap set conjecture*, obtaining an exponentially improved upper bound for subsets of $\mathbb{F}_3^n$ containing no three distinct elements in arithmetic progression. We discussed this result in an earlier blog post.

A reduction developed by Alon, Shpilka, and Umans (Alon et al. 2013) then allowed these polynomial-method results to establish the Erdős–Szemerédi sunflower conjecture in the case $r = 3$.

However, extending the result to $r = 4, 5, \dots$ proved difficult, and the general conjecture remained open.

The new paper of Bucić, He, and Lee (Bucić et al. 2026) settles the problem for every number of petals.

Theorem 1 (Bucić–He–Lee) *For every integer $r \geq 3$, there exists a constant $c_r < 2$ such that, for every integer $n \geq r - 1$ and every set X with $|X| = n$, any family $\mathcal{F} \subseteq 2^X$ containing no r distinct members that form a sunflower satisfies*

$$|\mathcal{F}| \leq c_r^n.$$

Thus, for every fixed $r \geq 3$, the proportion of subsets of an n-element set that can belong to a family without an r-sunflower is at most

$$\frac{|\mathcal{F}|}{2^n} \leq \left(\frac{c_r}{2}\right)^n.$$

Since $c_r/2 < 1$, this proportion tends to zero exponentially fast as n grows.

In fact, the authors obtain an explicit estimate:

$$|\mathcal{F}| \leq 3\left(2 - \frac{1}{16r^3}\right)^n.$$

The constant 3 in front does not affect the exponential conclusion. The authors do not attempt to optimize the constant in the exponent.

Let us briefly discuss the proof.

The central idea is to translate the sunflower problem into a problem about vectors.

Suppose that the ground set is divided into several disjoint blocks, and that every set in a certain family contains exactly one element from each block.

We can then represent each set by a vector

$$(x_1, x_2, \ldots, x_m),$$

where the coordinate x_i specifies which element was selected from the ith block.

In this representation, r sets form a sunflower precisely when, in every coordinate, their entries are either all equal or all distinct.

For example, the vectors

$$(1,2), \qquad (1,3), \qquad (1,4)$$

form a 3-sunflower: the first coordinates are all equal, while the second coordinates are all distinct.

This turns the original problem about intersections of sets into a problem about coordinatewise equality patterns.

The authors develop a polynomial method to bound the number of vectors that can be chosen without creating such a pattern. They construct a polynomial expression that detects sunflower configurations and use a dimension-counting argument to show that a sunflower-free family cannot be too large.

The key innovation is an interpolation argument that works for an arbitrary number r of vectors. Earlier polynomial methods were particularly effective for three-element configurations, but extending them to larger numbers of elements was a major obstacle.

A second ingredient converts the resulting vector bound back into a statement about arbitrary families of subsets. This uses a probabilistic selection of blocks and an averaging argument to control the number of sunflower-free sets.

Combining the two steps gives the required exponential saving.

It is important to emphasize what this theorem does *not* prove. The Erdős–Rado sunflower conjecture remains open. That conjecture asks for an exponential bound depending only on the maximum cardinality w of the sets, regardless of how large their ground set is.

The new Erdős–Szemerédi theorem instead gives an exponential bound in the size n of the ground set. These are different parameters, and one result does not imply the other.

Nevertheless, the theorem represents substantial progress in understanding sunflower configurations. It extends the polynomial method far beyond the three-petal case and resolves a conjecture that had remained open since the 1970s.

Its conclusion is simple and striking: if we select more than c_r^n distinct subsets of an n-element set, where $c_r < 2$ is a suitable constant, then some r of the selected sets must have exactly the same pairwise intersection.

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