Two Closed Orbits on Almost Every Energy Surface

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On November 5, 2026, the *Annals of Mathematics* published a paper by Rohil Prasad entitled *High-dimensional families of holomorphic curves and three-dimensional energy surfaces* (Prasad 2026). The paper, first posted as a preprint in 2024, establishes two remarkable results about Hamiltonian dynamical systems. It shows that every compact regular energy surface in four-dimensional phase space contains infinitely many proper compact invariant subsets, and that almost every such surface contains at least two closed orbits.

Let us recall the problem.

Consider a mechanical system with two position coordinates q_1, q_2 and two momentum coordinates p_1, p_2. Its state at any moment is represented by a point

$$x = (q_1, q_2, p_1, p_2) \in \mathbb{R}^4.$$

Let $H : \mathbb{R}^4 \to \mathbb{R}$ be an infinitely differentiable function, called the *Hamiltonian*. In classical mechanics, H usually represents the total energy of the system.

The motion is described by *Hamilton's equations*

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad i = 1, 2,$$

where the dot denotes differentiation with respect to time.

Given an initial state x, these equations determine its trajectory. We denote by $\phi^t(x)$ the state reached after time t. The family of transformations ϕ^t is called the *Hamiltonian flow*.

One of the fundamental properties of Hamiltonian systems is that energy is conserved:

$$H(\phi^t(x)) = H(x).$$

Consequently, if the initial state satisfies $H(x) = E$, then the entire trajectory remains in the *energy surface*

$$Y_E = \{x \in \mathbb{R}^4 : H(x) = E\}.$$

We will consider energy surfaces that are *compact*, meaning closed and bounded, and *regular*, meaning that the gradient $\nabla H(x)$ is nonzero at every point of Y_E. Under these conditions, a nonempty Y_E is a smooth three-dimensional surface in $\mathbb{R}^4$.

A natural question is whether such a surface must contain a *closed orbit*, that is, a trajectory that returns to its starting point after some positive time T:

$$\phi^T(x) = x.$$

In this case the motion repeats periodically.

The famous *almost-existence theorem* states that almost every compact regular energy surface contains at least one closed orbit (Struwe 1990). Here “almost every” means that the exceptional set of energy values E has Lebesgue measure zero.

However, this leaves two interesting questions. What can we say about the exceptional energy surfaces? And can we guarantee more than one closed orbit on a typical energy surface?

To formulate the first question, we need one more definition.

A subset $\Lambda \subseteq Y_E$ is called *invariant* if every trajectory starting in Λ stays in Λ for all time:

$$\phi^t(\Lambda) = \Lambda \qquad \text{for every } t \in \mathbb{R}.$$

A nonempty invariant subset is called *proper* if $\Lambda \neq Y_E$.

Every closed orbit is an example of a compact invariant subset. However, an invariant subset need not consist of a periodic trajectory. It may contain many nonperiodic trajectories, and these trajectories can have extremely complicated behaviour.

In 1998, Herman (Herman 1998) asked whether every compact regular energy surface must contain a proper compact invariant subset, even if it has no closed orbits.

This question was answered affirmatively in 2023 by Fish and Hofer (Fish and Hofer 2023). Their theorem established that every such energy surface contains at least one proper compact invariant subset.

Prasad’s new theorem considerably strengthens this result and also improves the classical almost-existence theorem.

Theorem 1 (Prasad) *Let $H : \mathbb{R}^4 \to \mathbb{R}$ be an infinitely differentiable function.*

1. *Every nonempty compact regular energy surface Y_E contains infinitely many pairwise distinct, proper, compact invariant subsets whose union is dense in Y_E.*
2. *For almost every energy value E for which Y_E is nonempty, compact, and regular, the surface Y_E contains at least two geometrically distinct closed orbits.*

The first assertion is much stronger than merely guaranteeing the existence of one proper invariant subset. It says that such subsets are distributed throughout the entire energy surface.

More precisely, the union of these subsets is *dense* in Y_E: every nonempty open region of Y_E contains a point belonging to at least one of them.

Thus, the dynamics on a compact regular energy surface always possesses a rich collection of smaller invariant structures. This holds for *every* such surface, not just for almost every one.

The second assertion guarantees at least two different periodic trajectories for almost every compact regular energy surface. The orbits are counted geometrically, so following the same periodic trajectory several times does not produce additional orbits.

Interestingly, the number two is best possible.

To see this, consider the Hamiltonian

$$H = q_1^2 + p_1^2 + \alpha(q_2^2 + p_2^2),$$

where $\alpha > 0$ is irrational.

For every $E > 0$, the energy surface $H = E$ is a three-dimensional ellipsoid. Hamilton's equations describe rotations in the coordinate planes (q_1, p_1) and (q_2, p_2), with angular velocities 2 and 2α, respectively.

Since the ratio of these velocities is irrational, a trajectory with nonzero motion in both planes can never be periodic. However, when the motion is confined to just one coordinate plane, it is periodic.

Consequently, every positive regular energy surface has *exactly two* geometrically distinct closed orbits, corresponding to the two coordinate circles.

Thus, Prasad's lower bound cannot be improved to three without additional assumptions.

Let us briefly discuss the proof.

Prasad uses techniques from *symplectic geometry*, a branch of mathematics that studies the geometric structures underlying Hamiltonian mechanics. A central role is played by *pseudoholomorphic curves*, which are two-dimensional surfaces satisfying certain differential equations analogous to the Cauchy–Riemann equations from complex analysis.

The main innovation is to study high-dimensional families of such curves and extract information about the Hamiltonian flow from their geometric properties.

These methods allow Prasad to construct many proper compact invariant subsets and show that their union is dense in every compact regular energy surface. They also provide quantitative information about periodic trajectories as the energy value varies, leading to the almost-everywhere existence of at least two closed orbits.

An important feature of the theorem is its generality. The energy surfaces are not assumed to be convex or to satisfy any additional geometric condition beyond compactness and regularity.

In a previous blog post, we discussed a related result on counting closed Reeb orbits on convex hypersurfaces. Prasad's theorem concerns a different, more general setting: arbitrary smooth Hamiltonian systems in four-dimensional phase space. In exchange for this generality, the assertion about two closed orbits applies to almost every energy surface, rather than to every one.

The theorem also illustrates an important distinction between periodic orbits and general invariant subsets. Even on energy surfaces for which the existence of a closed orbit cannot be guaranteed, the first part of the theorem ensures that proper compact invariant subsets are abundant.

Prasad's paper provides a substantial new understanding of Hamiltonian dynamics in four dimensions. It strengthens two classical existence results simultaneously and shows that even the most complicated smooth Hamiltonian flows must possess a remarkable amount of invariant structure.

References

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