The Three-Quarter Exponent for Self-Avoiding Walks

Bogdan Grechuk · 8 Nov 2026

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In September 2026, OpenAI released a preprint (OpenAI 2026) announcing a proof of the famous three-quarter exponent for self-avoiding walks on the honeycomb lattice. The result concerns a fundamental question in probability theory and statistical physics: *how far does a random path typically spread if it is forbidden from visiting the same point twice?*

Let us recall the problem.

Consider the *honeycomb lattice*, also called the hexagonal lattice, obtained by tiling the plane with congruent regular hexagons. We regard the vertices of these hexagons as the vertices of a graph, with two vertices joined by an edge whenever they are adjacent along a side of a hexagon.

A *self-avoiding walk* of length n is a sequence

$$\gamma = (v_0, v_1, \dots, v_n)$$

of $n+1$ distinct vertices such that v_i and v_{i+1} are joined by an edge for every $i = 0, \dots, n-1$.

In other words, the walk consists of n steps along lattice edges and is never allowed to visit a vertex twice.

Self-avoiding walks were introduced as mathematical models of polymer chains. In a polymer, different parts of the same molecule cannot occupy the same physical position, so ordinary random walks, which may repeatedly return to previously visited points, are not always appropriate.

In 1953, Flory developed a theory relating the spatial size of a polymer to its length. A central prediction of this theory, subsequently refined using methods from statistical physics, is that a typical two-dimensional self-avoiding walk of length n should spread over a distance of order

$$n^{3/4}.$$

This is quite different from an ordinary random walk, whose typical displacement has order $n^{1/2}$. The self-avoidance condition forces the path to spread out more rapidly.

However, proving the exponent $3/4$ rigorously has been extraordinarily difficult.

Let c_n denote the number of self-avoiding walks of length n on the honeycomb lattice, starting at a fixed vertex. It is known that the limit

$$\mu = \lim_{n \to \infty} c_n^{1/n}$$

exists. The constant μ is called the *connective constant* of the lattice.

In 1982, Nienhuis (Nienhuis 1982) used methods from theoretical physics to predict that

$$\mu = \sqrt{2 + \sqrt{2}}.$$

In 2012, Duminil-Copin and Smirnov (Duminil-Copin and Smirnov 2012) proved this formula rigorously, establishing one of the most celebrated results in the mathematical theory of self-avoiding walks.

But counting walks and understanding their spatial geometry are two different problems. Knowing how rapidly c_n grows does not tell us how far a typical walk travels.

To study the latter question, choose a walk uniformly at random from all c_n self-avoiding walks of length n starting at a fixed vertex. Thus, every such walk has probability $1/c_n$.

For a walk $\gamma = (v_0, \dots, v_n)$, define its *diameter* by

$$D(\gamma) = \max_{0 \le i, j \le n} |v_i - v_j|,$$

where $|v_i - v_j|$ denotes the usual Euclidean distance between the corresponding points in the plane. We assume that the lattice edges have length 1.

The diameter measures the maximum distance between any two vertices visited by the walk.

Notice that the diameter is not necessarily the same as the distance $|v_n - v_0|$ between its endpoints. A self-avoiding walk may travel a long way from its starting point and then return close to it without crossing itself.

Let D_n denote the diameter of a uniformly chosen self-avoiding walk of length n.

The prediction from statistical physics is that

$$D_n = n^{3/4+o(1)}$$

with high probability, meaning that for every fixed $\epsilon > 0$, the diameter should lie between $n^{3/4-\epsilon}$ and $n^{3/4+\epsilon}$ with probability tending to 1.

Before 2026, even establishing this exponent on the honeycomb lattice remained open.

An important recent advance was obtained by Krachun and Panagiotis (Krachun and Panagiotis 2026), who proved that a typical n-step self-avoiding walk on the honeycomb lattice remains within distance of order $n/\log n$ from its starting point, with high probability. This was a quantitative improvement over earlier results showing that the walk is *sub-ballistic*, meaning that its displacement grows more slowly than a constant multiple of n.

However, the predicted exponent $3/4$ was still out of reach.

The new OpenAI preprint (OpenAI 2026) claims to establish it.

Theorem 1 (OpenAI) *For every $\epsilon > 0$ and $A > 0$, there exists an integer n_0 such that, for every integer $n \geq n_0$,*

$$\mathbb{P}\left(n^{3/4-\epsilon} \leq D_n \leq n^{3/4+\epsilon}\right) \geq 1 - n^{-A},$$

where D_n is the diameter of a uniformly chosen n-step self-avoiding walk on the honeycomb lattice.

Thus, the theorem establishes the predicted exponent $3/4$, with a remarkably strong probability estimate.

Not only does the probability of an exceptional walk tend to zero, but it can be bounded by n^{-A} for *any* prescribed positive constant A, provided n is sufficiently large.

The result also gives more detailed information about the geometry of a typical walk.

Suppose that a walk of length n has diameter approximately $n^{3/4}$. Reversing this relationship suggests that a region of diameter R should contain approximately

$$R^{4/3}$$

vertices of the walk.

This predicts a geometric dimension of $4/3$.

In fact, the preprint proves a corresponding statement about the number of vertices visited inside disks, together with estimates for the number of smaller disks needed to cover the walk. These estimates hold simultaneously over a wide range of spatial scales, up to arbitrarily small powers of n.

The exponent $4/3$ also appears in the theory of *Schramm–Loewner evolution*, or SLE. Lawler, Schramm, and Werner (Lawler et al. 2004) proposed that the scaling limit of two-dimensional self-avoiding walks should be described by $\mathrm{SLE}_{8/3}$, a random fractal curve whose Hausdorff dimension is $4/3$.

The new theorem is consistent with this prediction, but it does not establish convergence of self-avoiding walks to $\mathrm{SLE}_{8/3}$.

Let us briefly discuss the proof.

The authors build on the exact value of the honeycomb connective constant and develop estimates for special self-avoiding walks called *bridges*. A bridge is, roughly speaking, a walk that crosses a strip of the lattice while satisfying restrictions on how it meets the strip’s boundaries.

Such bridges are useful because they can be joined together in a controlled way. The proof studies their lengths, the distances they cross, and how they can be combined without creating self-intersections.

The next step is to obtain estimates that prevent a typical walk from travelling too quickly through a large region or from accumulating too many vertices inside a small region. These estimates connect the number of steps taken by a walk to the spatial distances it covers.

A further argument converts estimates involving specially weighted families of walks into probability bounds for walks of every sufficiently large fixed length. This is important because the theorem concerns the uniform distribution on walks of exactly n steps, not an average over different lengths.

The proof uses several companion papers establishing the necessary estimates for bridges, polygons, and other geometric configurations on the honeycomb lattice.

There are important limitations to the conclusion. The theorem establishes the exponent for the *diameter* of the walk, but does not prove the same lower bound for the distance between its starting and ending points at every sufficiently large length. Nor does it establish an asymptotic formula of the form $Cn^{3/4}$ with a fixed constant C, or the corresponding result for the square lattice.

The preprint is therefore an important step towards, rather than a complete resolution of, the broader problem of understanding the scaling behaviour of two-dimensional self-avoiding walks.

Nevertheless, if its proof is confirmed, it will establish one of the central predictions of polymer physics: a typical long self-avoiding walk on the honeycomb lattice spreads over a distance proportional to the three-quarter power of its length, up to arbitrarily small errors in the exponent.

References

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