Julia Sets with Positive Area

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This post continues my series on favorite theorems of the twenty-first century. My choice for 2006 in Analysis is a theorem of Xavier Buff and Arnaud Chéritat on the geometry of *Julia sets*. Their paper first appeared as a preprint in May 2006 and was subsequently published in the *Annals of Mathematics* in 2012 (Buff and Chéritat 2012).

The theorem answers a question that goes back to the founders of complex dynamics, Pierre Fatou and Gaston Julia: *can the Julia set of a quadratic polynomial have positive area?*

Let us begin with the definitions.

Consider a polynomial $f : \mathbb{C} \to \mathbb{C}$, where $\mathbb{C}$ denotes the set of complex numbers. Given a starting point $z_0 \in \mathbb{C}$, we repeatedly apply f:

$$z_1 = f(z_0), \qquad z_2 = f(z_1), \qquad z_3 = f(z_2), \quad \dots$$

The resulting sequence is called the *orbit* of z_0. We write

$$f^n(z_0) = \underbrace{f(f(\cdots f(z_0)\cdots))}_{n \text{ times}}.$$

One of the fundamental questions in complex dynamics is to understand which orbits remain bounded and which escape to infinity.

For a polynomial f of degree at least two, the set

$$K_f = \{z \in \mathbb{C} : \text{the sequence } f^n(z) \text{ is bounded}\}$$

is called the *filled Julia set*. Its boundary

$$J_f = \partial K_f$$

is called the *Julia set*.

Intuitively, the Julia set separates different kinds of dynamical behaviour. Points close to it may have orbits that behave very differently, even if their starting positions differ by an arbitrarily small amount.

For example, consider the simplest quadratic polynomial

$$f(z) = z^2.$$

Then

$$f^n(z) = z^{2^n}.$$

If $|z| < 1$, the orbit converges to 0. If $|z| > 1$, it escapes to infinity. If $|z| = 1$, it remains on the unit circle.

Consequently,

$$K_f = \{z : |z| \leq 1\}, \qquad J_f = \{z : |z| = 1\}.$$

In this example, the filled Julia set is the closed unit disk, while the Julia set is just its boundary, the unit circle.

The two sets have very different areas. The filled Julia set has area π, whereas the Julia set has area zero.

For more complicated quadratic polynomials, Julia sets can have extraordinarily intricate shapes. Even the family

$$f_c(z) = z^2 + c, \qquad c \in \mathbb{C},$$

produces a remarkable variety of fractal boundaries, closely connected with the famous Mandelbrot set.

But how large can these fractals be?

We measure their size using *Lebesgue measure*, which is the usual notion of area in the plane. A line or a smooth curve has area zero, even though it contains infinitely many points. More complicated fractals can also have area zero.

In fact, the Julia set of every quadratic polynomial has empty interior: it does not contain any open disk. However, a set with empty interior can still have positive area. Such sets are sometimes informally called *fat fractals*.

For many familiar quadratic polynomials, their Julia sets were known to have area zero. In particular, Petersen and Zakeri (Petersen and Zakeri 2004) proved in 2004 that, for almost every complex number c satisfying $|c| = 1$, the Julia set of

$$f(z) = z^2 + cz$$

has area zero and is *locally connected*. The latter property means, roughly speaking, that arbitrarily small neighbourhoods of points in the Julia set contain connected neighbourhoods within the set.

Here “almost every” refers to the usual measure on the unit circle of possible values of c. The theorem allows exceptional values of c, but these form a set of measure zero.

Could some of these exceptional polynomials have Julia sets of positive area?

Buff and Chéritat answered this question affirmatively.

Theorem 1 (Buff–Chéritat) *There exist quadratic polynomials whose Julia sets have positive Lebesgue measure.*

This is remarkable because a Julia set can have positive area despite having empty interior. It is also a much stronger conclusion than saying that its Hausdorff dimension equals 2: a set can have full Hausdorff dimension and still have area zero.

In fact, Buff and Chéritat proved something stronger. They constructed three different types of quadratic polynomials with Julia sets of positive area: examples with a *Cremer fixed point*, examples with a *Siegel disk*, and examples admitting infinitely many special renormalizations.

Let us briefly explain the ideas behind their construction.

Consider the quadratic polynomial

$$P_\theta(z) = e^{2\pi i\theta} z + z^2,$$

where θ is an irrational real number. The point $z = 0$ is a fixed point, since $P_\theta(0) = 0$, and

$$P'_\theta(0) = e^{2\pi i\theta}.$$

Thus, near the origin, the polynomial resembles a rotation through the angle $2\pi\theta$.

For some values of θ, this resemblance is exact after a suitable analytic change of coordinates. In this case, the polynomial has a *Siegel disk*: an open region around the origin on which its dynamics is equivalent to an irrational rotation.

A Siegel disk has positive area because it is open. But it belongs to the interior of the filled Julia set, not to the Julia set itself. Thus, merely constructing a Siegel disk does not answer our question.

The central idea of Buff and Chéritat was to make very small changes to the parameter θ, creating increasingly complicated dynamics while preserving almost all of the area of the filled Julia set.

More precisely, their construction produces a sequence of parameters $\theta_1, \theta_2, \ldots$ for which, at each step,

$$\operatorname{area}(K_{P_{\theta_{n+1}}}) \geq (1 - \epsilon_n) \operatorname{area}(K_{P_{\theta_n}}),$$

where the positive numbers ϵ_n can be chosen so that

$$\prod_{n=1}^{\infty} (1 - \epsilon_n) > 0.$$

At the same time, they arrange for periodic orbits to appear arbitrarily close to the fixed point 0.

The parameters converge to a limiting value θ, for which 0 is a *Cremer fixed point*: an irrationally rotating fixed point that cannot be transformed analytically into an ordinary rotation.

For this limiting quadratic polynomial, the filled Julia set has no interior, so it coincides with the Julia set. The careful preservation of area throughout the construction ensures that this Julia set has positive area.

Of course, establishing the required area estimates is highly nontrivial. The proof combines delicate perturbation arguments, renormalization techniques, and earlier results of McMullen, Yoccoz, Inou, and Shishikura.

The theorem does not contradict Petersen and Zakeri's earlier result. The parameters constructed by Buff and Chéritat are exceptional, while Petersen and Zakeri proved that Julia sets have area zero for almost every rotation parameter.

The result also leaves interesting questions about particular polynomials. Even when a quadratic polynomial is given by a precise formula, determining whether its Julia set has zero or positive area can be extremely difficult. A particularly important family arises from *Feigenbaum renormalization*, which plays a central role in the study of period-doubling phenomena in dynamical systems.

Buff and Chéritat's theorem changed our understanding of the possible geometry of Julia sets. It demonstrated that the boundary separating bounded and escaping dynamics can be much larger, in the sense of area, than previously known examples suggested.

A fractal boundary need not merely be infinitely complicated: it can occupy a positive proportion of a region of the plane, while still containing no open disk.

References

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