At Least Six Colours for the Plane

Bogdan Grechuk · 3 Nov 2026

https://functor.network/user/3333/entry/2046

In September 2026, OpenAI released a preprint (OpenAI 2026) announcing a major breakthrough on the famous *chromatic number of the plane* problem. The authors claim to prove that at least six colours are required to colour the plane so that no two points at distance exactly 1 have the same colour.

The problem was originally posed by Edward Nelson in 1950 and is now known as the *Hadwiger–Nelson problem*. Its statement is remarkably simple.

Suppose that we want to colour every point of the Euclidean plane using a finite number of colours. The only restriction is that any two points at distance exactly 1 from each other must receive different colours.

What is the smallest number of colours sufficient to accomplish this?

This minimum is called the *chromatic number of the plane*, denoted by $\chi(\mathbb{R}^2)$.

Clearly, at least three colours are needed, because the three vertices of an equilateral triangle with side length 1 must all have different colours. A slightly more sophisticated argument, using a special arrangement of equilateral triangles, shows that even three colours do not suffice.

In 1961, Leo and William Moser (Moser and Moser 1961) gave a particularly elegant example of a finite configuration of points that cannot be properly coloured with three colours. Their construction, known as the *Moser spindle*, consists of just seven points.

On the other hand, John Isbell observed in 1950 that seven colours are sufficient. His construction uses a tiling of the plane by small regular hexagons, assigning colours periodically so that points in different hexagons of the same colour are always more than unit distance apart. The hexagons are chosen sufficiently small that two points in the same hexagon are less than unit distance apart.

Thus, already in 1950, mathematicians knew that

$$4 \leq \chi(\mathbb{R}^2) \leq 7.$$

Remarkably, these bounds remained unchanged for almost seventy years.

In 2018, Aubrey de Grey (de Grey 2018) made the first substantial breakthrough. He constructed a finite set of points in the plane such that every colouring of these points with four colours necessarily gives the same colour to two points at distance 1.

More precisely, de Grey constructed a finite graph whose vertices represent points in the plane and whose edges join pairs of points at distance exactly 1. Such a graph is called a *unit-distance graph*.

The *chromatic number* of a graph is the minimum number of colours needed to colour its vertices so that every pair of vertices joined by an edge have different colours. De Grey proved that a certain finite unit-distance graph has chromatic number at least 5.

This immediately implies that the entire plane cannot be coloured with four colours, because any colouring of the plane would induce a colouring of this finite graph.

Hence, de Grey's theorem improved the bounds to

$$5 \leq \chi(\mathbb{R}^2) \leq 7.$$

The next question was whether five colours suffice. The new OpenAI preprint (OpenAI 2026) claims to answer this question negatively.

Theorem 1 (OpenAI) *Every colouring of $\mathbb{R}^2$ with five colours contains two points of the same colour at distance exactly 1 from each other. Consequently,*

$$6 \leq \chi(\mathbb{R}^2) \leq 7.$$

An important feature of this theorem is that it applies to *every* colouring of the plane. There are no restrictions on the shapes of the colour classes: they need not be connected, measurable, or have any other regularity properties.

Let us briefly discuss the proof of the new result.

Unlike de Grey's proof, the OpenAI paper does not begin by constructing an explicit finite unit-distance graph that requires six colours. Instead, it develops a completely different approach, combining ideas from geometry, probability, and the theory of dynamical systems.

The proof has two main steps.

First, the authors show that the existence of an arbitrary proper five-colouring would imply the existence of a special *measurable* five-colouring, in which violations of the unit-distance restriction are allowed only on an exceptional set of measure zero.

Roughly speaking, this means that if we choose a point and a direction at random, then the probability that the point and its unit-distance neighbour in that direction have the same colour is zero.

This reduction is highly nontrivial because the original colouring may be completely irregular. It uses averaging over translations and rotations of the plane, together with methods from harmonic analysis.

The second step proves that even such a weak measurable five-colouring cannot exist.

The authors examine which colours can occur near points on circles of radius 1 and study the geometric restrictions imposed by the unit-distance condition. They construct certain connected interfaces between colour regions and show that the possible arrangements of five colours lead to contradictions. Interestingly, the classical Moser spindle also appears in the final part of the argument.

Thus, the proof combines powerful modern techniques with geometric ideas going back to the earliest investigations of the problem.

The authors also provide a formalization of the main result in Lean, a computer proof assistant. This offers an additional way of checking the mathematical argument, although independent examination of the proof and its formal statement remains important.

There is another interesting consequence. A classical theorem of de Bruijn and Erdős (de Bruijn and Erdős 1951) states that an infinite graph can be coloured with k colours if and only if every finite subgraph can be coloured with k colours.

Consequently, the new theorem implies that there must exist a *finite* unit-distance graph in the plane that cannot be coloured with five colours. However, the proof does not construct such a graph explicitly or give a useful bound on its number of vertices.

The exact value of the chromatic number of the plane is still unknown. We now have only two possibilities:

$$\chi(\mathbb{R}^2) = 6 \qquad \text{or} \qquad \chi(\mathbb{R}^2) = 7.$$

If the new proof is confirmed, it will be the second major breakthrough on the Hadwiger–Nelson problem in less than a decade, after nearly seventy years without progress. The remaining question is equally simple to state: can the entire plane be coloured with six colours while avoiding monochromatic pairs at distance 1, or are seven colours necessary?

References

de Bruijn, N. G., and Paul Erdős. 1951. “A Colour Problem for Infinite Graphs and a Problem in the Theory of Relations.” *Indagationes Mathematicae* 13: 371–73. https://users.renyi.hu/~p_erdos/1951-01.pdf.

de Grey, Aubrey D. N. J. 2018. “The Chromatic Number of the Plane Is at Least 5.” *Geombinatorics* 28 (1): 18–31. https://arxiv.org/abs/1804.02385.

Moser, Leo, and William Moser. 1961. “Solution to Problem 10.” *Canadian Mathematical Bulletin* 4 (2): 187–89. https://doi.org/10.1017/S0008439500025765.

OpenAI. 2026. *The Euclidean Plane Is Not Five-Colorable*. OpenAI Math Release preprint. https://github.com/openai/math/blob/main/preprints/The-Euclidean-plane-is-not-five-colorable-September-23-2026/paper.pdf.