Erdős’s Reciprocal-Sum Conjecture

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In September 2026, OpenAI released a preprint (OpenAI 2026) announcing a proof of a famous conjecture of Erdős: every set of positive integers whose reciprocals have a divergent sum contains arbitrarily long arithmetic progressions. The paper proves a much stronger quantitative result, giving a dramatic improvement in the bounds associated with Szemerédi’s theorem.

Recall that an *arithmetic progression of length* k is a sequence

$$a,\ a+d,\ a+2d,\ \ldots,\ a+(k-1)d,$$

where a and d are positive integers. For example, $3, 7, 11, 15$ is an arithmetic progression of length 4.

In 1975, Szemerédi (Szemerédi 1975) proved one of the most celebrated theorems in combinatorics: every subset of the positive integers with positive upper density contains arithmetic progressions of every finite length.

Here a set A of positive integers has *positive upper density* if

$$\limsup_{N\to\infty} \frac{\#(A\cap\{1,\ldots,N\})}{N} > 0,$$

where $\#S$ denotes the number of elements of a finite set S, and $\limsup$ denotes the limit superior.

Szemerédi’s theorem is a remarkable generalization of Roth’s 1953 theorem (Roth 1953), which established the same conclusion for progressions of length 3.

However, Szemerédi’s theorem does not apply to sets of zero density. A famous example is the set of prime numbers, which has density zero, but still contains arbitrarily long arithmetic progressions, as Green and Tao proved in 2004. We discussed this result in a previous post.

In fact, the prime numbers satisfy a stronger condition: Euler proved that the sum of their reciprocals diverges,

$$\sum_{p\ \text{prime}} \frac{1}{p} = \infty.$$

This motivated Erdős to formulate a far-reaching conjecture (Erdős 1981): every set A of positive integers satisfying

$$\sum_{a \in A} \frac{1}{a} = \infty$$

contains arithmetic progressions of every finite length.

Notice that this assumption is much weaker than positive upper density. For example, the prime numbers satisfy it despite having zero density.

The conjecture attracted considerable attention, but proving it turned out to be extremely difficult.

To understand the problem quantitatively, let $r_k(N)$ denote the largest possible number of elements in a subset of

$$\{1, 2, \ldots, N\}$$

that contains no arithmetic progression of length k.

Szemerédi’s theorem is equivalent to the statement that, for every fixed $k \geq 3$,

$$\lim_{N \to \infty} \frac{r_k(N)}{N} = 0.$$

However, this gives no indication of how quickly the ratio approaches zero.

In 2020, Bloom and Sisask (Bloom and Sisask 2020) made a major breakthrough by proving that

$$r_3(N) \leq C \frac{N}{(\log N)^{1+c}}$$

for some positive constants C, c, where $\log$ denotes the natural logarithm. This was sufficient to settle Erdős’s conjecture for arithmetic progressions of length 3.

In 2023, Kelley and Meka (Kelley and Meka 2023) obtained the much stronger bound

$$r_3(N) \leq N \exp\Big(-c(\log N)^{1/12}\Big)$$

for some positive constant c and all sufficiently large N. Subsequent work improved the exponent further.

For longer progressions, progress was slower. In particular, in 2024, Leng, Sah, and Sawhney (Leng et al. 2024) proved that for every fixed $k \geq 5$,

$$r_k(N) \leq C_k N \exp\Big(-(\log \log N)^{c_k}\Big)$$

for some positive constants C_k, c_k.

However, such bounds were not strong enough to settle Erdős’s reciprocal-sum conjecture for progressions of arbitrary length.

The new OpenAI preprint (OpenAI 2026) claims to achieve the required improvement.

Theorem 1 (OpenAI) *For every integer $k \geq 3$, there exist constants $C_k, c_k, \epsilon_k > 0$ such that, for every integer $N \geq 2$,*

$$r_k(N) \leq C_k N \exp\bigl(-c_k(\log N)^{\epsilon_k}\bigr).$$

Consequently, every set A of positive integers satisfying

$$\sum_{a \in A} \frac{1}{a} = \infty$$

contains arithmetic progressions of every finite length.

The bound is significantly stronger than merely asserting that $r_k(N)/N$ tends to zero. Indeed, for every fixed $B > 0$,

$$\exp\bigl(-c_k(\log N)^{\epsilon_k}\bigr) = o\left(\frac{1}{(\log N)^B}\right),$$

where $f(N) = o(g(N))$ means that $f(N)/g(N) \to 0$ as $N \to \infty$.

Thus, for every fixed progression length k, the theorem gives a saving greater than any fixed power of $\log N$.

Let us explain why this proves Erdős's conjecture. Suppose that a set A contains no arithmetic progression of length k. Divide the positive integers into intervals

$$[1,2),\ [2,4),\ [4,8),\ [8,16), \dots.$$

The interval $[2^m, 2^{m+1})$ contains at most $r_k(2^m)$ elements of A, because otherwise, after subtracting $2^m - 1$ from all its elements, we would obtain a subset of $\{1, \dots, 2^m\}$ containing a progression of length k.

Consequently,

$$\begin{aligned}\sum_{a \in A} \frac{1}{a} &\leq \sum_{m=0}^{\infty} \frac{r_k(2^m)}{2^m} \\ &\leq 1 + C_k \sum_{m=1}^{\infty} \exp\bigl(-c_k(m \log 2)^{\epsilon_k}\bigr).\end{aligned}$$

But the last series converges, because its terms eventually decrease faster than $1/m^2$.

Therefore, any set containing no arithmetic progression of length k must have a *convergent* sum of reciprocals. This proves the contrapositive of Erdős's conjecture.

Notice that we can apply the same argument after deleting any finite number of elements from A. Therefore, the theorem actually implies that every set with divergent reciprocal sum contains *infinitely many* arithmetic progressions of each fixed length.

In particular, since the sum of the reciprocals of the prime numbers diverges, the theorem provides another route to the famous Green–Tao theorem on arbitrarily long arithmetic progressions in primes.

The OpenAI paper remains a preprint, and independent verification of its quantitative estimates is important. If confirmed, it will resolve a celebrated conjecture of Erdős and establish a powerful new quantitative version of Szemerédi's theorem.

What makes the result particularly striking is the simplicity of the final conclusion: no matter how irregularly an infinite set of positive integers is distributed, if the sum of its reciprocals diverges, then it must contain arithmetic progressions of every finite length.

References

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