Short Egyptian Fractions

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On September 25, 2026, OpenAI released a preprint (OpenAI 2026) announcing a proof of a conjecture of Erdős concerning *Egyptian fractions*. The conjecture asks how many fractions with numerator 1 are needed to represent an arbitrary rational number. The result determines the answer up to a constant factor.

An *Egyptian fraction* is a representation of a positive rational number as a sum of distinct *unit fractions*, that is, fractions of the form $1/d$ with d a positive integer. Such representations were used in ancient Egyptian mathematics, hence their name.

For example,

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4},$$

while

$$\frac{4}{13} = \frac{1}{4} + \frac{1}{18} + \frac{1}{468}.$$

Notice that the denominators must be distinct, and that even a relatively simple fraction may require surprisingly large denominators.

It is a classical fact that every rational number strictly between 0 and 1 can be represented as an Egyptian fraction. One way to construct such a representation is the *greedy algorithm*: at each step, subtract the largest unit fraction that does not exceed the remaining number, and repeat.

More precisely, given a fraction a/b with $0 < a < b$, choose

$$d = \left\lceil \frac{b}{a} \right\rceil,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Then

$$\frac{a}{b} = \frac{1}{d} + \frac{ad - b}{bd}.$$

The numerator of the remaining fraction satisfies

$$0 \le ad - b < a.$$

Thus, repeating the procedure eventually gives a representation by distinct unit fractions.

However, this does not answer a more difficult question: *how many terms are actually necessary?*

For integers $1 \le a < b$, let $L(a,b)$ denote the smallest positive integer ℓ for which

$$\frac{a}{b} = \frac{1}{d_1} + \cdots + \frac{1}{d_\ell}, \qquad 2 \le d_1 < \cdots < d_\ell,$$

where all the denominators are integers.

Next, define

$$L(b) = \max_{1 \le a < b} L(a,b).$$

Thus, $L(b)$ is the largest number of terms that may be necessary when representing fractions with denominator b. We do not require a and b to be relatively prime, and there is no restriction on the sizes of the denominators d_i.

In 1950, Erdős (Erdős 1950) investigated this problem and proved that $L(b)$ grows at least as fast as a positive constant times $\log\log b$. Here $\log$ denotes the natural logarithm, so $\log\log b = \log(\log b)$. The lower bound already follows by considering the fractions

$$\frac{b-1}{b}.$$

Erdős conjectured that this lower bound gives the correct order of growth. In other words, he predicted that

$$L(b) = O(\log\log b).$$

The notation $O(\log\log b)$ means that $L(b) \le C \log\log b$ for some constant $C > 0$ and all sufficiently large b.

Proving this turned out to be extremely difficult. In 1985, Vose (Vose 1985) established the upper bound

$$L(b) = O(\sqrt{\log b}),$$

but the conjectured double-logarithmic bound remained out of reach.

The new OpenAI preprint (OpenAI 2026) claims to resolve the problem completely, up to the values of the constant factors.

Theorem 1 (OpenAI) *There exist absolute constants $c, C > 0$ and b_0 such that, for every integer $b \ge b_0$,*

$$c \log\log b \le L(b) \le C \log\log b.$$

Thus, no matter which numerator $1 \le a < b$ we choose, the fraction a/b can always be represented using at most $C \log\log b$ distinct unit fractions.

Moreover, the lower bound shows that this order of growth cannot be improved. For every sufficiently large b, there is at least one numerator a for which a representation requires at least $c \log \log b$ terms.

Notice that this is a statement about the *number of terms*, not the sizes of the denominators. The denominators in the resulting representations may be enormous. In fact, allowing very large denominators is essential to the method.

Let us briefly discuss the proof. The authors start by applying the greedy algorithm a limited number of times, leaving a fraction with a specially controlled denominator. They then construct a large integer with many divisors and show that a large proportion of fractions with this auxiliary denominator have very short Egyptian-fraction representations.

The key idea is that if sufficiently many numerators admit such representations, then every desired numerator can be obtained, after suitable scaling, as the sum of two of them. Combining their representations then gives the required short expansion of the original fraction.

The difficult part is proving that the exceptional numerators, which do not yet have short representations, remain sufficiently rare throughout the construction. The authors accomplish this using estimates involving divisors of integers and their remainders upon division.

The preprint also proves further results about the number of distinct Egyptian-fraction representations of 1. The main theorem has been formalized in Lean, and Bloom and van Doorn (Bloom and van Doorn 2026) have subsequently written a shorter human exposition and simplification of the arguments.

The result illustrates how a question that is extremely easy to formulate can lead to deep mathematics. Although Egyptian fractions have been studied since antiquity, determining the optimal number of terms required to represent an arbitrary rational number remained open for decades. The new theorem gives a remarkably simple answer: the worst-case number of terms grows only as the logarithm of a logarithm.

References

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