Sylvester’s Conjecture on Rational Cubes

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In September 2026, Ashay Burungale and Ye Tian (Burungale and Tian 2026) posted a preprint announcing the completion of a conjecture traditionally attributed to James Joseph Sylvester in 1879. The conjecture concerns a simple question in number theory: which prime numbers can be expressed as the sum of two cubes of rational numbers?

For example,

$$7 = 2^3 + (-1)^3,$$

and

$$13 = \left(\frac{7}{3}\right)^3 + \left(\frac{2}{3}\right)^3.$$

Thus, both 7 and 13 are sums of two rational cubes.

Allowing rational numbers rather than just integers makes a substantial difference. Indeed, the cube of any integer is congruent to 0, 1, or -1 modulo 9. Consequently, the sum of two integer cubes can only be congruent to 0, 1, 2, 7, or 8 modulo 9. Since $13 \equiv 4 \pmod 9$, it cannot be expressed as the sum of two *integer* cubes, although the rational representation above is possible.

More generally, if we write two rational numbers with a common denominator,

$$x = \frac{a}{c}, \qquad y = \frac{b}{c},$$

then the equation $x^3 + y^3 = p$ is equivalent to

$$a^3 + b^3 = pc^3,$$

where a, b, c are integers and $c \neq 0$.

Thus, our problem can be reformulated as follows: for which primes p does the equation

$$a^3 + b^3 = pc^3$$

have a solution in integers with $c \neq 0$?

Sylvester studied closely related equations in 1879. The conjecture traditionally associated with his name predicts that every prime congruent to 4, 7, or 8 modulo 9 is a sum of two rational cubes. Here $p \equiv 4 \pmod 9$, for example, means that the remainder when p is divided by 9 equals 4.

Interestingly, the exact historical attribution is not entirely clear. Dasgupta and Voight (Dasgupta and Voight 2009) noted that, although Sylvester investigated sums and differences of rational cubes, they were unable to locate this precise conjecture in his original writings.

For almost 150 years, proving the conjecture remained a major challenge.

In 1994, Elkies (Elkies 1994) announced a proof for primes $p \equiv 4, 7 \pmod 9$, but the complete details were not published. In 2018, Dasgupta and Voight (Dasgupta and Voight 2018) proved these two cases under the additional assumption that 3 is not a cube modulo p, that is, there is no integer t satisfying

$$t^3 \equiv 3 \pmod p.$$

Although this covered many primes, it did not establish the conjecture in full.

In May 2026, Hongbo Yin (Yin 2026) posted a preprint proving that every prime $p \equiv 4, 7 \pmod 9$ is a sum of two rational cubes, without any additional assumptions.

This left only the case $p \equiv 8 \pmod 9$. In September 2026, Burungale and Tian (Burungale and Tian 2026) proved this remaining case, completing the conjecture.

Theorem 1 (Yin; Burungale–Tian) *For every prime p satisfying*

$$p \equiv 4, 7, 8 \pmod 9,$$

there exist rational numbers x, y such that

$$x^3 + y^3 = p.$$

The conclusion is remarkable: a simple congruence condition on a prime number guarantees the existence of a rational solution to a cubic equation. The numerators and denominators of the rational numbers involved may be very large, and the theorem does not provide an elementary formula for finding them.

If the 2026 proofs are confirmed, they will settle a problem dating back almost a century and a half, providing another beautiful example of how the sophisticated theory of elliptic curves can resolve questions that are very easy to formulate in elementary number theory.

References

Burungale, Ashay, and Ye Tian. 2026. *A Proof of Sylvester's Conjecture*. https://arxiv.org/abs/2609.14893.

Dasgupta, Samit, and John Voight. 2009. “Heegner Points and Sylvester’s Conjecture.” In *Arithmetic Geometry*, vol. 8. Clay Mathematics Proceedings. American Mathematical Society. https://sites.math.duke.edu/~dasgupta/papers/clay.pdf.
Dasgupta, Samit, and John Voight. 2018. “Sylvester’s Problem and Mock Heegner Points.” *Proceedings of the American Mathematical Society* 146 (8): 3257–73. https://doi.org/10.1090/proc/14008.
Elkies, Noam D. 1994. “Heegner Point Computations.” *Algorithmic Number Theory*, Lecture notes in computer science, vol. 877: 122–33. https://doi.org/10.1007/3-540-58691-1_49.
Yin, Hongbo. 2026. *A Proof of the* $4, 7$ *Cases of Sylvester’s Conjecture on Cube Sums*. https://arxiv.org/abs/2605.25917.